SEMI-ALGEBRAS AND RINGS OF CONTINUOUS FUNCTIONS

by

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Thesis submitted for the degree of
Doctor of Philosophy

University of Edinburgh December, 1968

Preface

This thesis consists of two chapters which are independent in subject matter. However, the primary aim in both chapters is to investigate the relationship between a given topological space and an appropriate space of continuous functions defined on the topological space. Chapter I deals with ordered topological spaces and the semi-algebras of monotonic continuous real-valued functions defined on them. Chapter II deals with the F-spaces of Gillman and Henriksen and the rings, or algebras, of continuous complex-valued functions defined on them.

Each chapter consists of an introduction and sections numbered §u . In each section, propositions are marked u.v , where u is the section number, v the proposition number. There is no attempt to classify results into theorems, propositions and lemmas.

I am greatly indebted to my research supervisor Professor F.F. Bonsall for the constant guidance and encouragement received during the preparation of this thesis.

Contents

Symbols

Set theory

$X-S$	The complement of S in X.
$\mathrm{cl}_{\mathcal{T}} S$	The closure of S for the topology $\mathcal{T}$.
$\emptyset$	The empty set.

Other set-theoretic notation used is standard.

Chapter I

CHAPTER 1

SEMI-ALGEBRAS OF MONOTONIC FUNCTIONS

Introduction

Topological spaces on which an order relation is also defined were first studied by Nachbin beginning in the year 1947. His results are available in the book "Topology and Order" ([10]). With each ordered topological space is associated the semi-algebra consisting of the non-negative real-valued continuous monotonic increasing functions defined on the space. The systematic study of this semi-algebra was initiated by Bonsall in [2].

The relations between topological spaces and the rings of real-valued continuous functions which they carry, are dealt with in the book "Rings of continuous functions" by Gillman and Jerison ([7]). Our approach to the study of an ordered topological space and the semi-algebra it carries, is basically that which is adopted in this book, although on a much smaller scale.

In working with semi-algebras, one is unable to use the usual notion of a maximal ideal. The concept of a bi-ideal is introduced and, to a limited extent, this notion plays a role analogous to that of an ideal in a ring of continuous functions.

§1 contains the basic information on ordered topological spaces which is used in the text. Our main interest is in the treatment of an ordered topological space as a bitopological space, that is, a set on which two topologies are defined. §2 deals with the semi-algebra of monotonic functions which is associated with an ordered topological space. In §3, the notion of a bi-ideal in a pair of semi-algebras is introduced, and the basic information obtained. The structure space of maximal bi-ideals is defined in §4. For the case of a monotonically completely regular space, the structure space leads to a compactification for the space. This circle of ideas is developed in §5, §6 and §7. In §8 we introduce a notion of connectedness in an ordered topological space, and §9 discusses the interplay between order connectedness and the existence of idempotents in the associated semi-algebra.

§1 Ordered topological spaces

A quasi-order on a set X, denoted by $\leq$, is a reflexive, transitive relation on X. When $\leq$ is also anti-symmetric, it is called a partial order.

An ordered topological space is a triple $(X, \mathcal{T}, \leq)$, where X is a set, $\mathcal{T}$ is a topology on X, and $\leq$ is a quasi-order on X. When $\leq$ is a partial order, $(X, \mathcal{T}, \leq)$ is called a partially ordered topological space. The order $\leq$ is said to be closed if it is closed as a subset of $X \times X$ in the product topology. A subset A of X is said to be lower if $y \in A$ and $x \leq y$ implies $x \in A$. An upper set is defined dually. The set of all lower sets which are open is denoted by $\mathcal{L}$, and the set of all upper sets which are open is denoted by $\mathcal{U}$. It is easily seen that $\mathcal{L}$ and $\mathcal{U}$ are topologies on X, which are called the lower topology and upper topology, respectively, for $(X, \mathcal{T}, \leq)$. We say that $(X, \mathcal{T}, \leq)$ is monotonically T_1 if, given $x, y \in X$ with $x \not\leq y$, there exists $L \in \mathcal{L}$, $U \in \mathcal{U}$ such that $y \in L$, $x \notin L$, and $x \in U$, $y \notin U$. $(X, \mathcal{T}, \leq)$ is said to be monotonically separated if given $x, y \in X$ with $x \not\leq y$, there exists $L \in \mathcal{L}$, $U \in \mathcal{U}$ with $L \cap U = \phi$, and $y \in L$, $x \in U$. We say that $(X, \mathcal{T}, \leq)$ is convex if $\mathcal{T}$ has a sub-base consisting of the sets in $\mathcal{L}$ and $\mathcal{U}$, or equivalently, if every open set in $\mathcal{T}$ can be written as the intersection of an open upper set and an open lower set.

We adopt the following notation. If $\mathcal{T}_1$ and $\mathcal{T}_2$ are topologies on X, $\mathcal{T}_1 + \mathcal{T}_2$ is defined to be the coarsest topology on X containing both $\mathcal{T}_1$ and $\mathcal{T}_2$. With this notation, $(X, \mathcal{T}, \leq)$ is convex if and only if $\mathcal{T} = \mathcal{L} + \mathcal{U}$.

The most important ordered topological spaces turn out to be convex and monotonically separated. In this direction we have the following result.

<u>1.1</u> (Nachbin) Let $(X, \mathcal{T}, \leq)$ be a partially ordered topological space. If $\mathcal{T}$ is compact and $\leq$ is closed, then $(X, \mathcal{T}, \leq)$ is convex and monotonically separated.

For a proof of this result, see [10] p.46 (Theorem 4) and p.48 (Theorem 5).

Given $(X, \mathcal{T}, \leq)$ with lower and upper topologies $\mathcal{L}$ and $\mathcal{U}$, denote by $\leq_{\mathcal{L}}$ and $\leq_{\mathcal{U}}$ the relations on X defined by

$x \leq_{\mathcal{L}} y$ if and only if, given $L \in \mathcal{L}$, $y \in L \Rightarrow x \in L$.

$x \leq_{\mathcal{U}} y$ if and only if, given $U \in \mathcal{U}$, $x \in U \Rightarrow y \in U$.

Then it is easily seen that $\leq_{\mathcal{L}}$ and $\leq_{\mathcal{U}}$ are in fact quasi-orders on X .

<u>1.2</u> If $(X, \mathcal{T}, \leq)$ is monotonically T_1 , then $\leq$, $\leq_{\mathcal{L}}$ and $\leq_{\mathcal{U}}$ are identical.

<u>proof</u>: Let $x \leq y$. Then obviously $x \leq_{\mathcal{L}} y$ and $x \leq_{\mathcal{U}} y$. Suppose on the other hand that $x \not\leq y$. By the hypothesis there exists $L \in \mathcal{L}$, $U \in \mathcal{U}$ such that $y \in L$, $x \notin L$ and $x \in U$, $y \notin U$. This implies that $x \not\leq_{\mathcal{L}} y$ and $x \not\leq_{\mathcal{U}} y$, and hence the result.

<u>1.3</u> If $(X, \mathcal{T}, \leq)$ is monotonically T_1 and convex, then $(X, \mathcal{T}, \leq) = (X, \mathcal{L}+\mathcal{U}, \leq_{\mathcal{L}})$.

<u>proof</u>: The proof is immediate from 1.2 .

It follows that a convex, monotonically T_1 space is completely determined by its lower and upper topologies, and we can write $(X, \mathcal{T}, \leq)$ as a bi-topological space $(X, \mathcal{L}, \mathcal{U})$.

Bi-topological spaces were introduced and studied by Kelly [9]. There is a certain parallel between the theory of bi-topological spaces and the theory of ordered topological spaces. 1.3 gives a class of ordered topological spaces which can be treated as bi-topological spaces. The converse problem of determining a class of bi-topological spaces which can be treated as ordered topological spaces is left open.

By 1.1 , compact spaces with a closed partial order are convex and monotonically T_1 , and so are determined as bi-topological spaces. We give another important example.

Example The real line R with its natural topology and order is easily shown to be convex and monotonically separated.

Let $(X_1, \mathcal{T}_1, \leq_1)$ and $(X_2, \mathcal{T}_2, \leq_2)$ be ordered topological spaces. When there is no ambiguity, we use $\leq$ to denote the quasi-order in both X_1 and X_2. A mapping τ from X_1 into X_2 is said to be monotonic increasing if and only if

$$x , y \in X_1 \text{ and } x \leq y \Rightarrow \tau(x) \leq \tau(y) .$$

A monotonic decreasing mapping is defined dually.

The following result shows when a monotonic mapping can be considered as a mapping which is continuous in two pairs of topologies. For the case $X_2 = R$, this reduces to Theorem 7(iii) in [3] .

1.4 Let τ be a mapping from $(X_1, \mathcal{T}_1, \leq)$ into a monotonically T_1 and convex space $(X_2, \mathcal{T}_2, \leq)$. Let $\mathcal{L}_1, \mathcal{L}_2, \mathcal{U}_1, \mathcal{U}_2$ be the respective lower and upper topologies in X_1 and X_2. Then the following conditions are equivalent.

(a) τ is monotonic increasing and continuous in the topologies $\mathcal{T}_1$ and $\mathcal{T}_2$.

(b) τ is continuous in both pairs of topologies $\mathcal{L}_1, \mathcal{L}_2$ and $\mathcal{U}_1, \mathcal{U}_2$.

proof: Suppose (a) holds. Then the inverse image of an open lower set in X_2 is an open lower set in X_1, and dually for open upper sets. Thus (b) holds.

Now suppose that (b) holds. Let T be an open set in $\mathcal{T}_2$. By convexity, $T = L \cap U$ where $L \in \mathcal{L}_2$, and $U \in \mathcal{U}_2$. From (b), $\tau^{-1}(L)$ and $\tau^{-1}(U) \in \mathcal{T}_1$. Hence

$$\tau^{-1}(T) = \tau^{-1}(L \cap U) = \tau^{-1}(L) \cap \tau^{-1}(U) \in \mathcal{T}_1 .$$

Therefore, τ is continuous in the topologies $\mathcal{T}_1$ and $\mathcal{T}_2$. To show that τ is monotonic increasing, suppose that $x, y \in X_1$ and $\tau(x) \not\leq \tau(y)$. Since X_2 is monotonically T_1, there exists $L \in \mathcal{L}_2$ such that $\tau(y) \in L$ but $\tau(x) \notin L$. From (b), $\tau^{-1}(L) \in \mathcal{L}_1$. Also, $y \in \tau^{-1}(L)$ but $x \notin \tau^{-1}(L)$. Hence $x \not\leq y$. Therefore, (a) holds.

§2 Semi-algebras of monotonic functions

Given an ordered topological space $(X, \mathcal{T}, \leq)$, we define the following sets of functions.

$$C_R(X) = \{ f ; f \text{ is a continuous real-valued function on } X \} .$$

$$C_R^+(X) = \{ f \in C_R(X) : x \in X \Rightarrow f(x) \geq 0 \} .$$

$$C_R^+(X, \leq) = \{ f \in C_R^+(X) : x,y \in X \text{ and } x \leq y \Rightarrow f(x) \leq f(y) \} .$$

$$C_R^+(X, \geq) = \{ f \in C_R^+(X) : x,y \in X \text{ and } x \leq y \Rightarrow f(x) \geq f(y) \} .$$

When there is no confusion as to which ordered topological space is referred to, we write for the last two sets

$$A^{\uparrow} = C_R^+(X, \leq) ,$$

$$A^{\downarrow} = C_R^+(X, \geq) .$$

A <u>semi-algebra</u> (of continuous functions) is a subset A of $C_R(X)$ such that

$$f , g \in A \Rightarrow f + g \in A ,$$

$$\lambda \geq 0 \text{ and } f \in A \Rightarrow \lambda f \in A ,$$

$$f , g \in A \Rightarrow fg \in A .$$

All the sets defined above are easily seen to be semi-algebras. $A^{\uparrow}$ and $A^{\downarrow}$ are referred to as semi-algebras of monotonic increasing, respectively decreasing, real-valued functions.

We shall have occasion to refer to the lattice operations on $C_R(X)$. Given $f , g \in C_R(X)$ define

$$(f \vee g)(x) = \max \{ f(x), g(x) \} ,$$

$$(f \wedge g)(x) = \min \{ f(x), g(x) \} .$$

The functions $f \vee g$ and $f \wedge g$ so defined are continuous.

$A^{\uparrow}$, $A^{\downarrow}$ (and $C_R^+(X)$) are stable under the lattice operations, that is,

$$f\,,\ g \in A^{\uparrow} \Rightarrow f \vee g \text{ and } f \wedge g \in A^{\uparrow}.$$

Moreover, $A^{\uparrow}$ and $A^{\downarrow}$ are <u>reductive</u> in the following sense.

$$f \in A^{\uparrow}\,,\quad \lambda \geq 0 \Rightarrow (f - \lambda) \vee 0 \in A^{\uparrow}.$$

By a <u>closed semi-algebra</u> we mean a semi-algebra which is a closed subset of $C_R(X)$ in the topology of uniform convergence on compacta.

A semi-algebra A is <u>type-1</u> if it satisfies the condition

$$f \in A \Rightarrow \frac{f}{1+f} \in A.$$

The semi-algebras $A^{\uparrow}$ and $A^{\downarrow}$ are closed and type-1 . Moreover they contain the identity 1 and hence all non-negative constants. It was proved by Bonsall in [2] that the type-1 condition characterizes $A^{\uparrow}$ and $A^{\downarrow}$, in the following sense.

<u>2.1</u> Let $A \subseteq C^+_R(X)$ be a closed type-1 semi-algebra containing the unit function 1 . Let $\lesssim_A$ be the quasi-order on X defined by

$x \lesssim_A y$ if and only if, given $f \in A$, $f(x) \leq f(y)$.

Then, $\quad A = A^{\uparrow} = C^+_R(X, \lesssim_A)$.

Given $f \in C_R(X)$, the <u>zero set</u> $Z(f)$ of f is defined by

$$Z(f) = \{x \in X : f(x) = 0\}.$$

When $f \in A^{\uparrow}$, $Z(f)$ is a closed lower set and is called a <u>lower zero set</u> . An <u>upper zero set</u> is defined dually.

Note that if $Z(g)$ $(g \in A^{\downarrow})$ is an upper zero set, then setting $f = \left(\frac{2}{1+g} - 1\right) \vee 0$, we have $f \in A^{\uparrow}$ and $g(x) = 0$ if and only if $f(x) \geq 1$. Hence upper zero sets are precisely the sets of the form

$$\{x \in X : f(x) \geq 1\} \qquad (f \in A^{\uparrow}) .$$

Similarly, lower zero sets are precisely the sets of the form

$$\{x \in X : g(x) \geq 1\} \qquad (g \in A^{\downarrow}) .$$

The family of all lower zero sets is denoted by $\mathfrak{Z}^{\downarrow}$ and the family of all upper zero sets by $\mathfrak{Z}^{\uparrow}$.

An elementary fact about zero sets is the following.

<u>2.2</u> $\mathfrak{Z}^{\downarrow}$ and $\mathfrak{Z}^{\uparrow}$ are closed under finite unions and countable intersections.

<u>proof</u>: (a) Let f_1 , $f_2 \in A^{\uparrow}$. Then $f_1 f_2 \in A^{\uparrow}$ and $Z(f_1) \cup Z(f_2) = Z(f_1 f_2)$.

(b) Let $\{f_n\}$ be a sequence in $A^{\uparrow}$. Since $Z(f_n \wedge 1) = Z(f_n)$ we can assume that $\|f\| \leq 1$.

Then
$$f = \sum_{n=1}^{\infty} 2^{-n} f_n \quad \text{is in } A^{\uparrow} \text{, and}$$
$$Z(f) = \bigcap_{n=1}^{\infty} Z(f_n) \quad .$$

Similar proofs hold for $\mathfrak{Z}^{\uparrow}$.

<u>§ 3</u> <u>Ideals and bi-ideals</u>

Let $A \subseteq C_R(X)$ be a semi-algebra of continuous functions containing 1 . An <u>ideal</u> of A is a subset I of A such that

$$f , g \in I , h \in A , \lambda \geq 0 \Rightarrow f + g , \lambda f , hf \in I .$$

I is proper if $I \neq A$. It is clear that I is proper if and only if $1 \notin I$. Henceforth we assume that all ideals considered are proper.

Ideals of $A^{\uparrow}$ have been studied by Page [11], who showed that for the compact case, each point of the ordered topological space is associated with a closed prime type-1 ideal.

That each point cannot be associated with an ideal in $A^{\uparrow}$ which is maximal with respect to set inclusion, is shown by the following example.

Example Let $(X, \mathcal{T}, \leq)$ be the unit interval $[0,1]$ with its usual topology and order. Given $p \in X$, let

$$I_p = \{f \in A^{\uparrow} : f(p) = 0\} .$$

Then I_p is an ideal of $A^{\uparrow}$. For all $p \in X$ we have

$$I_p \subseteq I_0 ,$$

and if $p \neq 0$, the inclusion is proper, so that $I_{\frac{1}{2}}$, for example, is not maximal.

The above situation is in contrast to that for the ring $C_R([0,1])$, where all ideals of the form

$$M_p = \left\{f \in C_R([0,1]) : f(p) = 0\right\} \quad (p \in [0,1]) ,$$

are maximal with respect to set inclusion.

We introduce the notion of a bi-ideal in a pair of semi-algebras.

Definition Let A and B be sub-semi-algebras of $C_R^+(X)$ /, and let I , J be ideals in A and B respectively. The pair (I,J) is said to be proper if and

we assume $1 \in A$, $1 \in B$.

only if, given $i \in I$, $j \in J$, the element $i + j$ is singular in $C_R^+(X)$. A bi-ideal in (A,B) is a pair (I,J) which is proper.

Remark There are other conceivable definitions of 'proper' pairs. For example, instead of requiring $i + j$ to be singular in $C_R^+(X)$, we could require it to be singular in the least semi-algebra containing both A and B. We choose the above form because of its application to $A^{\uparrow}$ and $A^{\downarrow}$.

Since $i + j$ is singular in $C_R^+(X)$ if and only if $Z(i + j) = Z(i) \cap Z(j) \neq \emptyset$, we have the following alternative definition of a bi-ideal.

Definition (alternative). With notation as before, (I,J) is a bi-ideal in (A,B) if and only if, given $i \in I$, $j \in J$, $Z(i) \cap Z(j) \neq \emptyset$.

Two bi-ideals (I_1,J_1) and (I_2,J_2) are said to be related by pairwise set inclusion if and only if $I_1 \subseteq I_2$ and $J_1 \subseteq J_2$. In this case we write

$$(I_1,J_1) \subseteq (I_2,J_2) .$$

3.1 Bi-ideals are inductively ordered by pairwise set inclusion.

The proof of this statement is standard and will be omitted.

It follows from Zorn's lemma that every bi-ideal is contained in a maximal (with respect to pairwise set inclusion) bi-ideal. We prove the following stronger statement on the existence of maximal bi-ideals.

<u>3.2</u> Let $\{f_\lambda : \lambda \in \Lambda\}$ and $\{g_\omega : \omega \in \Omega\}$ be subsets of A and B such that the family of zero-sets

$$\{Z(f_\lambda)\ ,\ Z(g_\omega)\ :\ \lambda \in \Lambda \text{ and } \omega \in \Omega\}$$

is non-empty, has non-empty members, and has the finite intersection property. Then there exists a maximal bi-ideal (M,N) for which, given $\lambda \in \Lambda$, $\omega \in \Omega$,

$$f_\lambda \in M \text{ and } g_\omega \in N .$$

<u>proof</u>: Define a subset I of A as follows. If $\Lambda = \emptyset$, let $I = (0)$. Otherwise, let

$$I = \{f \in A : Z(f) \supseteq \bigcap Z(f_\lambda) \text{ the intersection being finite}\} .$$

It is readily verified that I is an ideal in A. An ideal J of B is defined analagously.

It follows from the finite intersection property of the zero sets that (I,J) is a bi-ideal. Evidently, $f_\lambda \in I$, $g_\omega \in J$ for all $\lambda \in \Lambda$, $\omega \in \Omega$. Now by Zorn's lemma, (I,J) is contained in a maximal bi-ideal (M,N) which has the required properties.

The following characterization of maximal bi-ideals will be frequently used.

<u>3.3</u> Let (M,N) be a maximal bi-ideal and let $f \in A$. Then $f \in M$ if and only if given $m \in M$, $n \in N$, we have

$$Z(f) \cap Z(m) \cap Z(n) \neq \emptyset .$$

A dual statement holds for N .

proof: (a) Suppose $f \in M$. Given $m \in M$, $n \in N$, we have $f + m \in M$. Hence

$$Z(f) \cap Z(m) \cap Z(n) = Z(f + m) \cap Z(n) \neq \emptyset .$$

(b) Suppose $f \in A$ satisfies the given condition. Let (M,f) denote the ideal in A generated by M and f . Then (M,f) consists of elements of the form $m + af$ where $m \in M$, $a \in A$. Given $n \in N$,

$$\begin{aligned} Z(m + af) \cap Z(n) &= Z(m) \cap Z(af) \cap Z(n) \\ &\supseteq Z(f) \cap Z(m) \cap Z(n) \\ &\neq \emptyset . \end{aligned}$$

Hence $((M,f),N)$ is a bi-ideal containing (M,N) . By maximality, $(M,f) = M$, so that $f \in M$.

Corollary 1 Let (M,N) be a maximal bi-ideal and let $f_1 \in M$. If $f_2 \in A$ and $Z(f_2) \supseteq Z(f_1)$, then $f_2 \in M$.

proof: immediate from the above.

Corollary 2 Let (M,N) be a maximal bi-ideal and let f_1 , $f_2 \in A$. Then $f_1 + f_2 \in M$ if and only if $f_1 \in M$ and $f_2 \in M$.

proof: Since $Z(f_1 + f_2) \subseteq Z(f_1)$ and $Z(f_1 + f_2) \subseteq Z(f_2)$, by corollary 1 , $f_1 + f_2 \in M$ implies $f_1 \in M$ and $f_2 \in M$. Conversely, $f_1 \in M$ and $f_2 \in M$ implies $f_1 + f_2 \in M$ by definition.

An ideal I of A is said to be prime if and only if given f_1, $f_2 \in A$ and $f_1 f_2 \in I$, then either $f_1 \in I$ or $f_2 \in I$. A bi-ideal (I,J) is said to be bi-prime ~~pairwise prime~~ if and only if given $f \in A$, $g \in B$ and $fg = 0$, then either $f \in I$ or $g \in J$. The bi-ideal (I,J) is said to be prime if it is ~~pairwise prime~~ bi-prime and both I and J are prime in A and B respectively.

Just as maximal ideals in rings are prime, so maximal bi-ideals are prime.

3.4 Every maximal bi-ideal is prime.

proof: Let (M,N) be a maximal bi-ideal. To show that M is prime, let f_1, $f_2 \in A-M$. By 3.3, there exist m_1, $m_2 \in M$, n_1, $n_2 \in N$ such that

$$Z(f_1) \cap Z(m_1) \cap Z(n_1) = \emptyset ,$$
$$Z(f_2) \cap Z(m_2) \cap Z(n_2) = \emptyset .$$

It follows that

$$Z(f_1 f_2) \cap Z(m_1 + m_2) \cap Z(n_1 + n_2) = \emptyset .$$

so that $f_1 f_2 \notin M$. Thus M is prime. Similarly N is prime.

To show that (M,N) is bi-prime, let $f \in A-M$ and $g \in B-N$. Again by 3.3, there exist m_1, $m_2 \in M$, n_1, $n_2 \in N$ such that

$$Z(f) \cap Z(m_1) \cap Z(n_1) = \emptyset ,$$
$$Z(g) \cap Z(m_2) \cap Z(n_2) = \emptyset .$$

Since $Z(m_1) \cap Z(n_1) \cap Z(m_2) \cap Z(n_2)$

$$= Z(m_1 + m_2) \cap Z(n_1 + n_2) \neq \emptyset ,$$

we must have $Z(f) \cup Z(g) \neq X$, and so $fg \neq 0$. Thus (M,N) is bi-prime.

§ 4 The structure space

In this section, A and B again denote sub-semi-algebras of $C_R^+(X)$. Our aim is to topologize and order the set of maximal bi-ideals in (A,B) .

Let $\mathfrak{M}$ denote the set of all maximal bi-ideals in (A,B) . Given $f \in A$, and $g \in B$, define

$$F_f = \{(M,N) \in \mathfrak{M} : f \in M\} ,$$
$$G_g = \{(M,N) \in \mathfrak{M} : g \in N\} .$$

4.1 Topologies $\mathfrak{P}$ and $\mathfrak{Q}$ can be defined on $\mathfrak{M}$ by taking as a base for the closed sets, all sets of the form G_g and F_f respectively.

proof: We show that the family of sets of the form F_f $(f \in A)$, is closed under finite unions. Let f_1 , $f_2 \in A$. Then $f_1 f_2 \in A$. By 3.4 , every $(M,N) \in \mathfrak{M}$ is prime. Hence

$$\begin{aligned} (M,N) \in F_{f_1 f_2} &\Leftrightarrow f_1 f_2 \in M , \\ &\Leftrightarrow f_1 \in M \text{ or } f_2 \in M , \\ &\Leftrightarrow (M,N) \in F_{f_1} \cup F_{f_2} . \end{aligned}$$

Therefore $F_{f_1} \cup F_{f_2} = F_{f_1 f_2}$. Closure under finite unions is proved by finite induction. Similarly, all sets of the form G_g $(g \in B)$ are closed under finite unions.

Remark : The topologies $\mathfrak{P}$ and $\mathfrak{Q}$ can also be defined by the "hull-kernel" method.

A partial ordering is defined on $\mathcal{M}$ as follows. Let $\leq$ be the relation on $\mathcal{M}$ determined by $(M_1,N_1) \leq (M_2,N_2)$ if and only if $M_1 \supseteq M_2$ and $N_1 \subseteq N_2$. Then it is easy to see that $\leq$ is indeed a partial ordering on $\mathcal{M}$.

Definition: The <u>structure space</u> for (A,B) is the partially ordered topological space

$$(\mathcal{M}, \mathcal{S}, \leq) ,$$

where $\mathcal{S} = \mathcal{P} + \mathcal{Q}$ and $\leq$ is the partial ordering on $\mathcal{M}$ defined above.

<u>4.2</u> Given $f \in A$, $g \in B$, then F_f is a (closed) lower set and G_g is a (closed) upper set in the space $(\mathcal{M}, \mathcal{S}, \leq)$.

<u>proof</u>: F_f and G_g are closed by definition. We show that F_f is lower. Let $(M_2,N_2) \in F_f$ and let $(M_1,N_1) \leq (M_2,N_2)$. Then $M_1 \supseteq M_2$. Since $f \in M_2$, we have $f \in M_1$, so that $(M_1,N_1) \in F_f$. Hence F_f is lower. Similarly G_g is upper.

<u>4.3</u> The topological space $(\mathcal{M}, \mathcal{S})$ is a T_1-space.

<u>proof</u>: Suppose $(M_1,N_1) \neq (M_2,N_2)$. We can assume without loss of generality that $M_1 \not\subseteq M_2$. There exists $m_1 \in M_1$ such that $m_1 \notin M_2$. Thus $(M_1,N_1) \in F_{m_1}$ but $(M_2,N_2) \notin F_{m_1}$. Therefore $\mathcal{M}-F_{m_1}$ is a $\mathcal{Q}$-neighbourhood of (M_2,N_2) which does not contain (M_1,N_1).

Next, if $M_2 \not\subseteq M_1$, there exists $m_2 \in M_2$ such that $m_2 \notin M_1$. Proceeding as above, we obtain a $\mathcal{Q}$-neighbourhood of (M_1,N_1) which does not contain (M_2,N_2).

If $M_2 \subsetneq M_1$, then since (M_1,N_1) and (M_2,N_2) are distinct maximal bi-ideals, we must have $N_2 \nsubseteq N_1$. There exists $n \in N_2$ such that $n \notin N_1$. Then $(M_2,N_2) \in G_n$ but $(M_1,N_1) \notin G_n$. Hence $\mathcal{M}\text{-}G_n$ is a $\mathcal{P}$-neighbourhood of (M_1,N_1) which does not contain (M_2,N_2).

Thus each of (M_1,N_1) and (M_2,N_2) has a neighbourhood which does not contain the other, and $(\mathcal{M},\mathcal{S})$ is a T_1-space.

Remark: The (Jacobson) structure space for the ring of continuous functions on an arbitrary topological space is Hausdorff. It appears that $(\mathcal{M},\mathcal{S})$ is not always Hausdorff, so that our theory differs here from the analagous theory for rings of continuous functions.

The next result is essentially a lemma towards proving that the structure space is compact.

4.4 Let f , $f' \in A$ and g , $g' \in B$. Then

(i) $Z(f) \cap Z(f') = \emptyset \Rightarrow F_f \cap F_{f'} = \emptyset$,

(ii) $Z(g) \cap Z(g') = \emptyset \Rightarrow G_g \cap G_{g'} = \emptyset$,

(iii) $Z(f) \cap Z(g) = \emptyset \Rightarrow F_f \cap G_g = \emptyset$.

proof: (i) Suppose $(M,N) \in F_f \cap F_{f'}$. Then $f \in M$ and $f' \in M$ and hence $f + f' \in M$. From the definition of a bi-ideal, $Z(f + f') = Z(f) \cap Z(f') \neq \emptyset$.

(ii) is proved similarly.

(iii) Suppose $(M,N) \in F_f \cap G_g$. Then $f \in M$ and $g \in N$. From the definition of a bi-ideal, $Z(f) \cap Z(g) \neq \emptyset$.

<u>4.5</u> The structure space $(\mathfrak{M}, \mathcal{S}, \leq)$ is compact.

<u>proof</u>: $\mathcal{S}$ has a closed sub-base consisting of sets of the form F_f and G_g ($f \in A$ and $g \in B$) . Let

$$\left\{ F_{f_\lambda} , G_{g_\omega} : \lambda \in \Lambda \text{ and } \omega \in \Omega \right\}$$

be a family of these sets with the finite intersection property. It follows from 4.4 that the family of zero-sets

$$\left\{ Z(f_\lambda) , Z(g_\omega) : \lambda \in \Lambda \text{ and } \omega \in \Omega \right\}$$

has the finite intersection property. By 3.2 , there exists a maximal bi-ideal (M,N) such that for all $\lambda \in \Lambda$, $\omega \in \Omega$, we have

$$f_\lambda \in M \quad \text{and} \quad g_\omega \in N .$$

This means that $(M,N) \in F_{f_\lambda}$, $(M,N) \in G_{g_\omega}$.

Therefore

$$(M,N) \in \bigcap \left\{ F_{f_\lambda} , G_{g_\omega} : \lambda \in \Lambda , \omega \in \Omega \right\} .$$

Compactness now follows from a theorem in general topology. See, for example, [14] p.112 Theorem F .

<u>4.6</u> Let $f , f' \in A$, $g , g' \in B$. Then

(i) $F_f \cap F_{f'} = F_{f+f'}$,

(ii) $G_g \cap G_{g'} = G_{g+g'}$,

(iii) if $Z(f) \cup Z(g) = X$, then $F_f \cup G_g = \mathfrak{M}$.

<u>proof</u>: $(M,N) \in F_{f+f'}$

$\Leftrightarrow$ $f + f' \in M$,

$\Leftrightarrow$ $f \in M$ and $f' \in M$ (3.3 corollary 2) ,

$\Leftrightarrow$ $(M,N) \in F_f \cap F_{f'}$.

Therefore $F_{f+f'} = F_f \cap F_{f'}$.

Similarly (ii) , $G_{g+g'} = G_g \cap G_{g'}$.

Now suppose that $Z(f) \cup Z(g) = X$, or equivalently, $fg = 0$. Then

$$\begin{aligned} (M,N) \in \mathcal{M} &\Rightarrow f \in M \text{ or } g \in N \quad (\text{ by } 3.4) , \\ &\Rightarrow (M,N) \in F_f \text{ or } (M,N) \in G_g , \\ &\Rightarrow (M,N) \in F_f \cup G_g . \end{aligned}$$

Hence $F_f \cup G_g = \mathcal{M}$.

§5 Monotonically completely regular spaces

An ordered topological space $(X, \mathcal{T}, \leq)$ is said to monotonically completely regular if and only if it is monotonically separated and

(i) If $U \in \mathcal{U}$ is an upper neighbourhood of a point $p \in X$, there exists $f \in A^{\uparrow}$ such that $0 \leq f \leq 1$ and $f(p) = 1$, $f(X-U) = 0$.

(ii) If $L \in \mathcal{L}$ is a lower neighbourhood of a point $p \in X$, there exists $g \in A^{\downarrow}$ such that $0 \leq g \leq 1$ and $g(p) = 1$, $g(X-L) = 0$.

This definition coincides with Nachbin's definition of a "uniformizable preordered space" in the case when $(X, \mathcal{T}, \leq)$ is convex. ([10] p.52) .

Note that in condition (i) above, the function $g = 1 - f$ has the property that $0 \leq g \leq 1$, $g(p) = 0$, $g(X-U) = 1$. A similar statement holds for (ii) . These equivalent formulations will be used interchangeably.

<u>5.1</u> (Nachbin). A compact monotonically separated ordered topological space is monotonically completely regular.

This is proved in [10] Theorem 7 p.55 .

Monotonically completely regular spaces play the same role in the study of $A^{\uparrow}$ and $A^{\downarrow}$ as completely regular spaces do in the study of the ring $C_R(X)$. We show in this section that we lose nothing in the study of the algebraic properties of $A^{\uparrow}$ and $A^{\downarrow}$ by assuming that $(X, \mathcal{T}, \leq)$ is monotonically completely regular.

Recall that $\mathfrak{Z}^{\downarrow}$ is the family of lower zero-sets and $\mathfrak{Z}^{\uparrow}$ is the family of upper zero-sets.

<u>5.2</u> Let $(X, \mathcal{T}, \leq)$ be monotonically separated. Then $(X, \mathcal{T}, \leq)$ is monotonically completely regular if and only if $\mathfrak{Z}^{\downarrow}$ is a closed base for $\mathcal{U}$ and $\mathfrak{Z}^{\uparrow}$ is a closed base for $\mathcal{L}$.

<u>proof</u>: (a) Suppose that $(X, \mathcal{T}, \leq)$ is monotonically completely regular. Let $U \in \mathcal{U}$ and $p \in U$. Then there exists $f \in A^{\uparrow}$ such that $f(p) = 1$, $f(X-U) = 0$. Hence $p \in X-Z(f) \subseteq U$ which shows that $\mathfrak{Z}^{\downarrow}$ is a closed base for $\mathcal{U}$ since $X-Z(f)$ is an open upper set given $f \in A^{\uparrow}$. Similarly $\mathfrak{Z}^{\uparrow}$ is a closed base for $\mathcal{L}$.

(b) Suppose that $\mathfrak{Z}^{\downarrow}$ and $\mathfrak{Z}^{\uparrow}$ are closed bases for $\mathcal{U}$ and $\mathcal{L}$ respectively. Let $U \in \mathcal{U}$, $p \in U$. By the hypothesis, there exists $f \in A^{\uparrow}$ such that $p \in$

$X-Z(f) \subseteq U$. Then $X-U \subseteq Z(f)$ and $f(p) = r > 0$.
Define

$$f' = \frac{f}{r} \wedge 1 .$$

Then $f' \in A^{\uparrow}$, $0 \leq f' \leq 1$, $f'(p) = 1$, $f'(X-U) = 0$.
Hence condition (i) of the definition of monotonic complete regularity is satisfied. Similarly, condition (ii) is satisfied.

Given $p \in X$, $f \in A^{\uparrow}$, we say that $Z(f)$ is a <u>lower zero-set neighbourhood</u> of p if $Z(f)$ contains an $\mathcal{L}$-neighbourhood of p . An <u>upper zero-set neighbourhood</u> is defined dually.

<u>5.3</u> Let $(X, \mathcal{T}, \leq)$ be monotonically completely regular and let $p \in L \in \mathcal{L}$. Then there exists $Z \in \mathcal{Z}^{\downarrow}$ and $Z' \in \mathcal{Z}^{\uparrow}$ such that

$$p \in X-Z' \subseteq Z \subseteq L .$$

A dual statement holds for $p \in U \in \mathcal{U}$.

<u>proof</u>: By monotonic complete regularity, there exists $g \in A^{\downarrow}$ such that $g(p) = 1$, $g(X-L) = 0$. Let

$$Z = \left\{x \in X : g(x) \geq \tfrac{1}{4} \right\} ,$$

$$Z' = \left\{x \in X : g(x) \leq \tfrac{3}{4} \right\} .$$

Then Z and Z' have the required properties

<u>Corollary</u> The lower zero-set neighbourhoods of a given point $p \in X$, form a fundamental system of neighbourhoods in $\mathcal{L}$. A dual statement holds for upper zero-set neighbourhoods.

The next two results give the relationship between monotonically completely regular topologies and weak topologies.

<u>5.4</u> Let $(X, \mathcal{T}, \leq)$ be convex and monotonically separated. Then $(X, \mathcal{T}, \leq)$ is monotonically completely regular if and only if $\mathcal{T}$ is the weak topology determined by either $A^{\uparrow}$ or $A^{\downarrow}$.

<u>proof</u>: Since the sets of the form $(-\infty, r]$ and $[r, \infty)$, $r \in R$, form a closed subbase for the topology on R, their preimages

$\{x \in X : f(x) \leq r\}$ and $\{x \in X : f(x) \geq r\}$ ($f \in A^{\uparrow}$, $r \in R$) form a closed subbase for the weak topology on X generated by $A^{\uparrow}$. We show that these sets are precisely the lower and upper zero-sets. Given $f \in A^{\uparrow}$, $r \in R$, let

$$f' = (f - r) \vee 0 \quad \text{and} \quad f'' = (f + 1 - r) \vee 0 .$$

Then f', $f'' \in A^{\uparrow}$ and

$$\{x \in X : f(x) \leq r\} = Z(f') ,$$
$$\{x \in X : f(x) \geq r\} = \{x \in X : f''(x) \geq 1\} .$$

Recalling the note on p.9, it follows from 5.2 and convexity of $\mathcal{T}$ that $\mathcal{T}$ is the weak topology on X determined by $A^{\uparrow}$.

We have seen that $A^{\downarrow}$ also determines $\mathfrak{z}^{\downarrow}$ and $\mathfrak{z}^{\uparrow}$. Repeating the argument, we find that $\mathcal{T}$ is also the weak topology determined by $A^{\downarrow}$.

This result can be sharpened to the following form.

<u>5.5</u> If $(X, \mathcal{T}, \leq)$ is monotonically separated and $\mathcal{T}$ is the weak topology on X determined by a family A of non-negative-valued increasing functions on X, then $(X, \mathcal{T}, \leq)$ is monotonically completely regular.

<u>proof</u>: Let $A^{\uparrow} = C_R^+(X, \mathcal{T}, \leq)$, and let $\mathcal{T}'$ be the weak topology on X determined by $A^{\uparrow}$.

Now $f \in A \Rightarrow f \in A^{\uparrow}$ so that $\mathcal{T}$ is coarser than $\mathcal{T}'$. But by definition $\mathcal{T}'$ is coarser than $\mathcal{T}$. Hence the two topologies are identical. By 5.4, $\mathcal{T} = \mathcal{T}'$ is monotonically completely regular.

We come now to the main result of this section, which allows us to make some strong assumptions on $(X, \mathcal{T}, \leq)$ for the purposes of studying $A^{\uparrow}$ and $A^{\downarrow}$.

<u>5.6</u> For each ordered topological space $(X, \mathcal{T}, \leq)$, there exist a partially ordered, convex, monotonically completely regular space $(X', \mathcal{T}', \leq')$ and a continuous monotonic increasing mapping α of X onto X', such that the mapping $h \to h \circ \alpha$ is an isomorphism of $A'^{\uparrow} = C_R^+(X', \mathcal{T}', \leq')$ onto $A^{\uparrow} = C_R^+(X, \mathcal{T}, \leq)$, and the mapping $k \to k \circ \alpha$ is an isomorphism of $A'^{\downarrow}$ onto $A^{\downarrow}$.

<u>proof</u>: Define $x \sim y$ in X to mean that $f(x) = f(y)$ for all $f \in A^{\uparrow}$. This is evidently an equivalence relation. Given $x \in X$, let $[x]$ denote the equivalence class to which x belongs. Let X' be the set of all equivalence classes $[x]$.

Define a relation $\leq'$ on X' as follows. Let

$[x] \leq' [y]$ if and only if $f(x) \leq f(y)$, for all $f \in A^{\uparrow}$. It is easy to show that this relation is well defined and is a partial ordering on X' .

Let α be the canonical mapping of X onto X' . Thus, given $x \in X$,

$$\alpha(x) = [x] \qquad (1) .$$

We have, easily, that α is monotonic increasing.

For each $f \in A^{\uparrow}$, define a mapping $h_f : X' \to R$ as follows. Given $[x] \in X'$, let $h_f[x] = f(x)$. Again it is easily checked that h_f is properly defined and is a non-negative real-valued monotonic increasing function. Further, distinct f give rise to distinct h_f and we have the identity

$$f = h_f \circ \alpha \qquad (2) .$$

Let $A = \{h_f : f \in A^{\uparrow}\}$, and let $\mathcal{T}'$ be the weak topology on X' generated by the family A .

It is evident that if $[x]$ and $[y]$ are points of X' such that $[x] \not\leq' [y]$, then there exists $h_f \in A$ such that $h_f[x] > h_f[y]$, which shows that $(X', \mathcal{T}', \leq')$ is monotonically separated. Now, by 5.5 , $(X', \mathcal{T}', \leq')$ is monotonically completely regular.

We show that α is continuous. The topology $\mathcal{T}'$ has a closed subbase consisting of sets of the form $h_f^{-1}[F]$, where $f \in A^{\uparrow}$ and F ranges over the closed subbase for R consisting of sets of the form $(-\infty, r]$, $[r, \infty)$, where r is real. Then

$$\begin{aligned} \alpha^{-1}\left[h_f^{-1}[F]\right] &= (h_f \circ \alpha)^{-1}[F] \\ &= f^{-1}[F] \qquad \text{by } (2) . \end{aligned}$$

By continuity of f, these sets are closed. This shows that α is continuous.

Next, we show that $A = A'^{\uparrow} = C_R^+(X', \mathcal{T}', \leq')$. Let $h' \in A'^{\uparrow}$. Since α is continuous and monotonic increasing, $h' \circ \alpha$ is continuous and monotonic increasing. Therefore $h' \circ \alpha \in A^{\uparrow}$. The identity (2) now shows that $h' \in A$. Therefore $A'^{\uparrow} \subseteq A$ and so $A'^{\uparrow} = A$.

Thus, the mapping $h \to h \circ \alpha$ maps $A'^{\uparrow}$ onto $A^{\uparrow}$. It is clear that this mapping preserves the semi-algebra operations and so is an isomorphism. This completes the first half of the theorem.

We begin the second half by showing that the space $(X', \mathcal{T}', \leq')$ can be constructed from the semi-algebra $A^{\downarrow}$. With notation as above, $x \sim y$ if and only if $g(x) = g(y)$ for all $g \in A^{\downarrow}$. This follows from the fact that $f \in A^{\uparrow} \Rightarrow \dfrac{1}{1+f} \in A^{\downarrow}$ and conversely $g \in A^{\downarrow} \Rightarrow \dfrac{1}{1+g} \in A^{\uparrow}$. These latter relations also show that $[x] \leq' [y]$ if and only if $g(x) \geq g(y)$ for all $g \in A^{\downarrow}$.

Given $g \in A^{\downarrow}$, define k_g by $k_g[x] = g(x)$. As before, we get $g = k_g \circ \alpha$. Let $B = \{k_g : g \in A^{\downarrow}\}$. Given a subbasic closed set F of R,

$$\begin{aligned} k_g^{-1}[F] &= \{[x] : k_g[x] = g(x) \in F\} \\ &= \alpha[g^{-1}[F]] \qquad , \end{aligned}$$

and similarly

$$h_f^{-1}[F] = \alpha[f^{-1}[F]] \quad .$$

We have seen on p.9 that the family of all sets $f^{-1}[F]$ coincides with the family of all sets $g^{-1}[F]$. Therefore the weak topology on X' generated by B, equals $\mathcal{T}'$, the weak topology on X' generated by A.

The rest of the proof now proceeds as in the first half, and we find that the mapping $k \to k \circ \alpha$ is an isomorphism of $A'^{\downarrow}$ onto $A^{\downarrow}$.

§ 6 Fixed bi-ideals and compact spaces

Throughout this and the next section, we assume that the space $(X, \mathcal{T}, \leq)$ is convex, partially ordered, and monotonically completely regular. Regarding this assumption, see 5.6. From now on, all bi-ideals are taken in $(A^{\uparrow}, A^{\downarrow})$.

A bi-ideal (I,J) is said to be <u>fixed</u> if there exists $p \in X$ such that

$$p \in \bigcap \left\{ Z(i) \, , \, Z(j) \, : \, i \in I \, , \, j \in J \right\} .$$

Such a point p is called a <u>fixing point</u> of the bi-ideal. If (I,J) is not fixed, it is said to be <u>free</u>.

Given $p \in X$, define

$$M_p^{\uparrow} = \left\{ f \in A^{\uparrow} : f(p) = 0 \right\} ,$$

$$M_p^{\downarrow} = \left\{ g \in A^{\downarrow} : g(p) = 0 \right\} .$$

Clearly, $(M_p^{\uparrow}, M_p^{\downarrow})$ is a fixed bi-ideal in $(A^{\uparrow}, A^{\downarrow})$ with fixing point p. With the above assumptions on $(X, \mathcal{T}, \leq)$, we have the following.

<u>6.1</u> The fixed maximal bi-ideals in $(A^{\uparrow}, A^{\downarrow})$ are precisely the pairs $(M_p^{\uparrow}, M_p^{\downarrow})$ for $p \in X$. These bi-ideals are distinct for distinct p.

<u>proof</u>: If (M,N) is a fixed maximal bi-ideal with fixing point p, then $(M,N) \subseteq (M_p^{\uparrow}, M_p^{\downarrow})$. Hence the only candidates for fixed maximal bi-ideals are those of the stated form.

Let $p \in X$, $f \in A^{\uparrow}$ and $f \notin M_p^{\uparrow}$. Then $f(p) = r > 0$. Let $U = \{x \in X : f(x) > \frac{1}{2}r\}$. Then $U \in \mathcal{U}$ and $p \in U$. By monotonic complete regularity, there exists $g \in A^{\downarrow}$ such that $g(p) = 0$, $g(X-U) = 1$. Hence $g \in M_p^{\downarrow}$, but since $Z(f) \cap Z(g) = \emptyset$, the pair $((M_p^{\uparrow}, f), M_p^{\downarrow})$ is not a bi-ideal. A dual result holds for $g \in A^{\downarrow}$, $g \notin M_p^{\downarrow}$. It follows that $(M_p^{\uparrow}, M_p^{\downarrow})$ is a maximal bi-ideal.

Suppose that p and q are distinct points of X. Since $\leq$ is a partial order, we can assume without loss of generality that $q \nleq p$. By monotonic complete regularity, we find an $f \in A^{\uparrow}$ such that $f(p) = 0$ and $f(q) = 1$. Hence $f \in M_p^{\uparrow}$ but $f \notin M_q^{\uparrow}$. Thus $(M_p^{\uparrow}, M_p^{\downarrow})$ and $(M_q^{\uparrow}, M_q^{\downarrow})$ are distinct.

<u>6.2</u> Let $(X, \mathcal{T}, \leq)$ be compact. Then every bi-ideal in $(A^{\uparrow}, A^{\downarrow})$ is fixed.

<u>proof</u>: Let (I,J) be a bi-ideal. The family of zero-sets

$$\{Z(i)\ ,\ Z(j)\ :\ i \in I\ ,\ j \in J\}$$

has the finite intersection property. By compactness, there exists a fixing point p of (I,J).

When $(X,\mathcal{T},\leq)$ is compact, it follows from 6.1 and 6.2 that every maximal bi-ideal is of the form $(M_p^{\uparrow},M_p^{\downarrow})$. In the non-compact case, there can exist free maximal bi-ideals as the following example shows.

Example Let $X = N$, the set of non-negative integers. Give X its discrete topology and natural order. $A^{\uparrow}$ is then the set of non-negative monotonic increasing sequences and $A^{\downarrow}$ is the set of non-negative monotonic decreasing sequences.

Let $I = (0)$,

$$J = \{g \in A^{\downarrow} : g(n) = 0 \text{ for all but a finite number of points } n \in N\}.$$

(I,J) is evidently a free bi-ideal. By Zorn's lemma, (I,J) is contained in a maximal bi-ideal which must also be free.

We have seen that a maximal bi-ideal can have at most one fixing point. The following is a sharpening of this statement.

6.3 If (I,J) is a bi-prime bi-ideal in $(A^{\uparrow},A^{\downarrow})$, then it has at most one fixing point.

proof: First, suppose that p is a fixing point of (I,J), and that $\mathbb{Z}(f)$ $(f \in A^{\uparrow})$ is a lower neighbourhood of p. We show that $f \in I$. By 5.3, there exists $g \in A^{\downarrow}$ such that $p \in X\text{-}Z(g) \subseteq Z(f)$. Hence $Z(f) \cup Z(g) = X$ and $fg = 0$. Since

$g(p) \neq 0$, $g \notin J$, so by the bi-prime condition, $f \in I$. Similarly, if $Z(g)$ is an upper neighbourhood of p , than $g \in J$.

Next, suppose that (I,J) has two distinct fixing points p and q . Since $\leq$ is a partial order, we can assume without loss of generality that $p \nleq q$. By monotonic complete regularity and 5.3 , we can find $f \in A^{\uparrow}$, $g \in A^{\downarrow}$ such that $Z(f)$ and $Z(g)$ are disjoint neighbourhoods of q and p respectively. By the first half, we have $f \in I$, $g \in J$, which contradicts the definition of a bi-ideal.

<u>Corollary</u>: If $(X, \mathcal{T}, \leq)$ is compact, every bi-prime bi-ideal has a unique fixing point.

<u>§7 A compactification for $(X, \mathcal{T}, \leq)$</u>

In this section, we show that $(X, \mathcal{T}, \leq)$ can be embedded as a dense subspace of $(\mathfrak{M}, \mathcal{S}, \leq)$ in such a way that every bounded function $f \in A^{\uparrow}$ has a continuous increasing extension over $\mathfrak{M}$.

From 6.1 , we see that the mapping $p \to (M_p^{\uparrow}, M_p^{\downarrow})$ is one to one from X into $\mathfrak{M}$. Accordingly, we can consider X as a subset of $\mathfrak{M}$ and we write $X \subseteq \mathfrak{M}$. If F is any subset of $\mathfrak{M}$, then $p \in X \cap F$ means that $(M_p^{\uparrow}, M_p^{\downarrow}) \in F$.

<u>7.1</u> $(X, \mathcal{T}, \leq)$ is (homeomorphic to) a subspace of $(\mathfrak{M}, \mathcal{S}, \leq)$.

<u>proof</u>: Given p , $q \in X$, we have by monotonic complete regularity

$$\begin{aligned} p \leq q \quad &\Leftrightarrow \quad M_p^{\uparrow} \supseteq M_q^{\uparrow} \quad \text{and} \quad M_p^{\downarrow} \subseteq M_q^{\downarrow} \ , \\ &\Leftrightarrow \quad (M_p^{\uparrow}, M_p^{\downarrow}) \leq (M_q^{\uparrow}, M_q^{\downarrow}) \ . \end{aligned}$$

Therefore the restriction of the order on $\mathfrak{M}$ to X , coincides with the given order on X .

Next we show that the relative topology of $\mathcal{P}$ on X coincides with $\mathcal{L}$. Let $g \in A^{\downarrow}$. Then

$$\begin{aligned} p \in G_g \cap X \quad &\Leftrightarrow \quad (M_p^{\uparrow}, M_p^{\downarrow}) \in G_g \ , \\ &\Leftrightarrow \quad g \in M_p^{\downarrow} \ , \\ &\Leftrightarrow \quad g(p) = 0 \ , \\ &\Leftrightarrow \quad p \in Z(g) \ . \end{aligned}$$

The identity map on X carries a family of basic closed sets for the relative topology of $\mathcal{P}$ on X , onto a family of basic closed sets for the topology $\mathcal{L}$. Therefore, the relative topology of $\mathcal{P}$ on X coincides with $\mathcal{L}$.

Similarly, the relative topology of $\mathcal{Q}$ on X coincides with $\mathcal{U}$. By convexity of the topologies $\mathcal{S}$ and $\mathcal{T}$, the relative topology of $\mathcal{S}$ on X coincides with $\mathcal{T}$.

Hence the mapping $p \to (M_p^{\uparrow}, M_p^{\downarrow})$ is a homeomorphism of $(X, \mathcal{T}, \leq)$ into a subspace of $(\mathfrak{M}, \mathcal{S}, \leq)$.

<u>7.2</u> Let $(X, \mathcal{T}, \leq)$ be compact. Then the structure space $(\mathfrak{M}, \mathcal{S}, \leq)$ is homeomorphic to $(X, \mathcal{T}, \leq)$.

<u>proof</u>: By 6.2 , the mapping $p \to (M_p^{\uparrow}, M_p^{\downarrow})$ is onto $\mathfrak{M}$. The result follows by 7.1 .

<u>Notation</u> Let F be a subset of $\mathfrak{M}$. Then $cl_{\mathcal{P}} F$ and $cl_{\mathcal{Q}} F$ denote the closures of F in $\mathfrak{M}$ with respect to the topologies $\mathcal{P}$ and $\mathcal{Q}$ respectively.

<u>7.3</u> Let $f \in A^{\uparrow}$, $g \in A^{\downarrow}$. Then $cl_{\mathcal{Q}} Z(f) = F_f$ and $cl_{\mathcal{P}} Z(g) = G_g$. In particular, setting $f = g = 0$, $cl_{\mathcal{P}} X = cl_{\mathcal{Q}} X = \mathfrak{M}$, so that $(X, \mathcal{T}, \leq)$ is dense in $(\mathfrak{M}, \mathcal{S}, \leq)$.

<u>proof</u>: Since $Z(f) \subseteq F_f$ and F_f is $\mathcal{Q}$-closed, we have $cl_{\mathcal{Q}} Z(f) \subseteq F_f$. On the other hand, suppose that $F_{f'} \supseteq Z(f)$ for some $f' \in A^{\uparrow}$. Then

$$Z(f') = X \cap F_{f'} \supseteq Z(f) .$$

Given $(M,N) \in \mathfrak{M}$, corollary 1 of 3.3 shows that $f \in M \Rightarrow f' \in M$. Hence $F_{f'} \supseteq F_f$. It follows that $cl_{\mathcal{Q}} Z(f) = F_f$.

Similarly, $cl_{\mathcal{P}} Z(g) = G_g$.

For the remainder of this section, τ will denote a continuous monotonic increasing mapping from $(X, \mathcal{T}, \leq)$ into a second ordered topological space $(Y, \mathcal{K}, \leq)$. Let $B^{\uparrow} = C_R^+(Y, \mathcal{K}, \leq)$, and $B^{\downarrow} = C_R^+(Y, \mathcal{K}, \geq)$.

Given $h \in B^{\uparrow}$, $k \in B^{\downarrow}$, we have $h \circ \tau \in A^{\uparrow}$ and $k \circ \tau \in A^{\downarrow}$. For $h \circ \tau$ and $k \circ \tau$ are certainly continuous. That they are monotonic increasing and decreasing respectively is also easily verified.

Let (I,J) be a bi-ideal in $(A^{\uparrow}, A^{\downarrow})$. Define

$$\tau^{\#}[I] = \{ h \in B^{\uparrow} : h \circ \tau \in I \} ,$$
$$\tau^{\#}[J] = \{ k \in B^{\downarrow} : k \circ \tau \in J \} .$$

7.4 With notation as above,

(a) $(\tau^{\#}[I], \tau^{\#}[J])$ is a bi-ideal in $(B^{\uparrow}, B^{\downarrow})$,

(b) if (I,J) is bi-prime, then so is $(\tau^{\#}[I], \tau^{\#}[J])$.

proof: (a) Let $h \in B^{\uparrow}$, h_1, $h_2 \in \tau^{\#}[I]$, $\lambda \geq 0$. The following relations show that $\tau^{\#}[I]$ is an ideal in $B^{\uparrow}$.

$$(h_1 + h_2) \circ \tau = (h_1 \circ \tau) + (h_2 \circ \tau) \quad ,$$
$$(hh_1) \circ \tau = (h \circ \tau)(h_1 \circ \tau) \quad ,$$
$$(\lambda h_1) \circ \tau = \lambda (h_1 \circ \tau) \quad .$$

Similarly, $\tau^{\#}[J]$ is an ideal in $B^{\downarrow}$.

Let $h \in \tau^{\#}[I]$, $k \in \tau^{\#}[J]$. Then

$$h \circ \tau \in I \text{ and } k \circ \tau \in J \Rightarrow Z(h \circ \tau) \cap Z(k \circ \tau) \neq \emptyset$$
$$\Rightarrow Z(h) \cap Z(k) \neq \emptyset \quad .$$

Hence $(\tau^{\#}[I], \tau^{\#}[J])$ is a bi-ideal.

(b) Suppose that (I,J) is bi-prime, $h \in B^{\uparrow}$, $k \in B^{\downarrow}$. Then

$$hk = 0 \Rightarrow (h \circ \tau)(k \circ \tau) = 0$$
$$\Rightarrow h \circ \tau \in I \text{ or } k \circ \tau \in J$$
$$\Rightarrow h \in \tau^{\#}[I] \text{ or } k \in \tau^{\#}[J] \quad .$$

Hence $(\tau^{\#}[I], \tau^{\#}[J])$ is bi-prime.

The next result gives the sense in which $(\mathfrak{M}, \mathcal{S}, \leq)$ is a compactification for $(X, \mathcal{T}, \leq)$.

7.5 Let $(Y, \mathcal{K}, \leq)$ be compact. Then each continuous monotonic increasing mapping of X into Y has a continuous monotonic increasing extension ~~of~~ $\mathfrak{M}$ into Y. mapping!

proof: Let τ be a continuous monotonic increasing mapping of X into Y. Given $(M,N) \in \mathfrak{M}$, (M,N) is bi-prime and by 7.4 , $(\tau^{\#}[M], \tau^{\#}[N])$ is bi-prime. Since Y is compact, the corollary to 6.3 shows that $(\tau^{\#}[M], \tau^{\#}[N])$ has a unique fixing point in Y . We define a mapping $\bar{\tau}$ of $\mathfrak{M}$ into Y as follows. Given $(M,N) \in \mathfrak{M}$, $\bar{\tau}(M,N)$ is the unique fixing point of $(\tau^{\#}[M], \tau^{\#}[N])$. We show that $\bar{\tau}$ is the required extension.

Given $p \in X$, $f(p) = 0$ for all $f \in M_p^{\uparrow}$. Therefore $\tau(p) \in Z(h)$ for all $h \in \tau^{\#}[M_p^{\uparrow}]$. Similarly, $\tau(p) \in Z(k)$ for all $k \in \tau^{\#}[M_p^{\downarrow}]$. Hence $\tau(p)$ is the unique fixing point of $(\tau^{\#}[M_p^{\uparrow}], \tau^{\#}[M_p^{\downarrow}])$, so $\bar{\tau}$ agrees with τ on X .

Fix $(M_o,N_o) \in \mathfrak{M}$. We next show that $\bar{\tau}$ is continuous at (M_o,N_o) in $\mathcal{P}$ and the lower topology on Y . Let $Z(h)$ $(h \in B^{\uparrow})$ be a lower zero-set neighbourhood of $\bar{\tau}(M_o,N_o)$. By 5.3 , there exists $k \in B^{\downarrow}$ such that

$$\bar{\tau}(M_o,N_o) \in Y-Z(k) \subseteq Z(h) ,$$

so that $Z(h) \cup Z(k) = Y$. Now $f = h \circ \tau \in A^{\uparrow}$ and $g = k \circ \tau \in A^{\downarrow}$, and $Z(f) \cup Z(g) = X$. By 4.6(iii) $F_f \cup G_g = \mathfrak{M}$. Since $\bar{\tau}(M_o,N_o) \notin Z(k)$, it follows that $(M_o,N_o) \notin G_g$. Therefore $\mathfrak{M}-G_g$ is a $\mathcal{P}$-neighbourhood of (M_o,N_o) . Moreover, for any $(M,N) \in \mathfrak{M}-G_g$, we have $(M,N) \in F_f$. Then $f = h \circ \tau \in M$, so that $h \in \tau^{\#}[M]$ and $\bar{\tau}(M,N) \in Z(h)$. This shows that $\bar{\tau}$ is continuous in $\mathcal{P}$ and the lower topology on Y .

Similarly, $\bar{\tau}$ is continuous in $\mathcal{Q}$ and the upper topology on Y . By convexity of $\mathcal{S}$ and $\mathcal{K}$, $\bar{\tau}$ is continuous in $\mathcal{S}$ and $\mathcal{K}$.

The proof that $\bar{\tau}$ is monotonic increasing is essentially the same as the proof in 1.4 . Suppose that $(M_1,N_1) \leq (M_2,N_2)$ and $\bar{\tau}(M_1,N_1) \not\leq \bar{\tau}(M_2,N_2)$. We can find a zero-set neighbourhood $Z(h)$ $(h \in B^{\uparrow})$ of $\bar{\tau}(M_2,N_2)$ which is disjoint from $\bar{\tau}(M_1,N_1)$. By continuity of $\bar{\tau}$ in $\mathcal{P}$ and the lower topology on Y , there exists $g \in A^{\downarrow}$ such that $(M_2,N_2) \in \mathfrak{M}\text{-}G_g$ and $(M,N) \in \mathfrak{M}\text{-}G_g \Rightarrow \bar{\tau}(M,N) \in Z(h)$. By 4.2 , G_g is an upper set so that

$$(M_1,N_1) \leq (M_2,N_2) \Rightarrow (M_1,N_1) \in \mathfrak{M}\text{-}G_g$$
$$\Rightarrow \bar{\tau}(M_1,N_1) \in Z(h) \quad ,$$

which is a contradiction. It follows that $\bar{\tau}$ is monotonic increasing. This completes the proof.

Let $A_b^{\uparrow}$ denote the subset of $A^{\uparrow}$ consisting of the bounded functions in $A^{\uparrow}$. Given $f \in A_b^{\uparrow}$, f is an increasing continuous function of X into a compact subspace of R , for example, the closure of the range of f . By 7.5 , f has a continuous increasing extension to $(\mathfrak{M}, \mathcal{S}, \leq)$, which we denote by $\bar{f}$.

That the extension $\bar{f}$ is unique follows from the density of $(X, \mathcal{T}, \leq)$ in $(\mathfrak{M}, \mathcal{S}, \leq)$ and a well known theorem of general topology. Hence there is a one to one correspondence between elements of $A_b^{\uparrow}$ and those of $C_R^+(\mathfrak{M}, \mathcal{S}, \leq)$, which also preserves the semi-algebra operations. This gives the following.

7.6 $A_b^{\uparrow}$ is isomorphic to $C_R^+(\mathfrak{M}, \mathcal{S}, \leq)$.

Remark For the case of the discrete order on X , the construction in 7.5 yields the Stone-Čech compactification for a completely regular space. For in this case $A^{\uparrow} = A^{\downarrow}$ and it is possible to prove that $(\mathfrak{M}, \mathcal{S})$ is Hausdorff.

§ 8 Order connectedness

In this section, $(X, \mathcal{T}, \leq)$ denotes an arbitrary ordered topological space, unless it is stated otherwise.

$(X, \mathcal{T}, \leq)$ is said to be order disconnected if and only if X can be written as a disjoint union $X = L \cup U$ of non-empty sets $L \in \mathcal{L}$, $U \in \mathcal{U}$. Such a pair (L,U) is said to be an order disconnection of X . $(X, \mathcal{T}, \leq)$ is said to be order connected if and only if it is not order disconnected.

Remark: These definitions are generalizations of those for connectedness in a topological space. Recently Pervin in [12] , has given the definitions for an arbitrary bi-topological space and several of our results were obtained by him. For this reason, we give only a selection of results in the theory of order connectedness, and without proofs. The only parts of this section needed for the next section are the definitions of disconnectedness.

8.1 A subspace of the real line R (with its natural topology and order) is order connected if and only if it is an interval. In particular, R is order connected.

Corollary: A subspace of R is connected if and only if it is order connected.

8.2 Any monotonic continuous image of an order connected space is order connected.

8.3 If $(X, \mathcal{T}, \leq)$ is order connected and $f \in A^{\uparrow}$, then $f(X)$ is an interval.

This last result is an Intermediate Value Theorem for monotonic functions.

Example Let $X = \{(x,y) \in R^2 : y \neq 0\}$. Let $\mathcal{T}$ be the relative topology on X as a subset of R^2, and define an ordering on X as follows.

$$(x_1,y_1) \leq (x_2,y_2) \text{ if and only if } x_1 \leq x_2 .$$

Then X is disconnected in the usual sense but it is order connected. The Intermediate Value Theorem breaks down for $f(x,y) = y$ $((x,y) \in X)$, but by 8.3, the theorem holds for functions $f \in A^{\uparrow}$.

8.4 $(X, \mathcal{T}, \leq)$ is order disconnected if and only if there exists $f \in A^{\uparrow}$ which maps X onto the set $\{0,1\}$.

$(X, \mathcal{T}, \leq)$ is said to be <u>totally order disconnected</u> if and only if given $x, y \in X$ with $x \not\leq y$, there exists an order disconnection (L, U) with $y \in L$, $x \in U$.

<u>8.5</u> Let $(X, \mathcal{T}, \leq)$ be monotonically separated. If $\mathcal{L}$ has an open base whose sets are also closed in $\mathcal{T}$, then $(X, \mathcal{T}, \leq)$ is totally order disconnected. A dual statement holds for $\mathcal{U}$.

In the compact case, the above has a converse.

<u>8.6</u> Let $(X, \mathcal{T}, \leq)$ be compact. If $(X, \mathcal{T}, \leq)$ is totally order disconnected, then $\mathcal{L}$ has an open base whose sets are also closed in $\mathcal{T}$, and $\mathcal{U}$ has an open base whose sets are also closed in $\mathcal{T}$.

<u>§9 Idempotents in $A^{\dagger}$</u>

Throughout this section, we assume that $(X, \mathcal{T}, \leq)$ is monotonically completely regular.

An element e of $A^{\dagger}$ is said to be an <u>idempotent</u> if and only if $e^2 = e$.

The existence of idempotents in $A^{\dagger}$ is related to order connectedness of $(X, \mathcal{T}, \leq)$, the relations being given in 9.1 and 9.2. A stronger property than total order disconnectedness is extreme order disconnectedness (defined later). This property is characterized in 9.7.

9.1 $(X, \mathcal{T}, \leq)$ is order connected if and only if 0 and 1 are the only idempotents in $A^{\uparrow}$.

proof: If e is an idempotent of $A^{\uparrow}$, the only values it can take are 0 and 1. The result follows from 8.4.

9.2 $(X, \mathcal{T}, \leq)$ is totally order disconnected if and only if $A^{\uparrow}$ is generated by its idempotents, that is, if and only if $A^{\uparrow}$ is the least closed type-1 semi-algebra containing these idempotents.

proof: (a) Suppose that $(X, \mathcal{T}, \leq)$ is totally order disconnected. Let A be the closed type-1 semi-algebra generated by the idempotents in $A^{\uparrow}$. Evidently

e $A \subseteq A^{\uparrow}$. By the Stone-Weierstrass theorem for type-1 semi-algebras (2.1),

$$A = C_R^+(X, \mathcal{T}, \leq_A) .$$

Hence it is sufficient to prove that the quasi-orders $\leq$ and $\leq_A$ are identical.

Clearly $x \leq y \Rightarrow x \leq_A y$. Suppose that $x \not\leq y$. By the hypothesis, there exists an order disconnection (L,U) with $y \in L$, $x \in U$. Let e be the function which takes the value 0 on L and the value 1 on U. Then $e \in A^{\uparrow}$, $e(x) \not\leq e(y)$. Thus, $\leq$ coincides with $\leq_A$.

(b) Suppose that $A^{\uparrow}$ is generated by its idempotents, and that $x_0 \not\leq y_0$. By monotonic complete regularity, there exists $f \in A^{\uparrow}$ such that $f(y_0) = 0$ and $f(x_0) = 1$. Since f can be uniformly approximated on the compact set $\{x_0, y_0\}$ by linear combinations of idempotents, there exists an idempotent $e \in A^{\uparrow}$ such that $e(y_0) = 0$ and $e(x_0) = 1$. Then setting

$$L = \{x \in X : e(x) = 0\} \quad , \quad U = \{x \in X : e(x) = 1\} ,$$

(L, U) is an ordered disconnection of X with $y_0 \in L$, $x_0 \in U$.

The next few results on semi-continuous functions are well known. Upper semi-continuity of a real-valued function is equivalent to continuity in the lower topology on R, and lower semi-continuity is equivalent to continuity in the upper topology on R.

Let $A_{\mathcal{L}}$ denote the set of non-negative upper semi-continuous functions defined on the topological space $(X, \mathcal{L})$, and let $A_{\mathcal{U}}$ denote the set of non-negative lower semi-continuous functions on the topological space $(X, \mathcal{U})$. By 1.4, we have

$$A_{\mathcal{L}} \cap A_{\mathcal{U}} = A^{\uparrow} .$$

<u>9.3</u> Let F be a bounded subset of $A_{\mathcal{U}}$ and let g be defined by

$$g(x) = \sup\{f(x) : f \in F\} < \infty .$$

Then $g \in A_{\mathcal{U}}$.

<u>proof</u>: Given $x_0 \in X$, $\varepsilon > 0$, there exists $f \in F$ such that

$$f(x_0) > g(x_0) - \tfrac{1}{2}\varepsilon .$$

Since $f \in A_{\mathcal{U}}$, there exists $U \in \mathcal{U}$ such that $x_0 \in U$ and $x \in U \Rightarrow f(x) > f(x_0) - \tfrac{1}{2}\varepsilon$.

Given $x \in U$,

$$g(x) \geq f(x) > f(x_o) - \tfrac{1}{2}\varepsilon > g(x_o) - \varepsilon .$$

Hence $g \in A_{\mathcal{U}}$.

<u>9.4</u> If $g \in A_{\mathcal{U}}$, then

$$g = \sup\{f \in A^{\uparrow} : f \leq g\} .$$

<u>proof</u>: Let $h = \sup\{f \in A^{\uparrow} : f \leq g\}$. Then $h \leq g$. Take $x_o \in X$. If $g(x_o) = 0$ then $h(x_o) = g(x_o) = 0$. Suppose $g(x_o) > 0$. Take δ such that $0 \leq \delta < g(x_o)$. Since $g \in A_{\mathcal{U}}$, there exists $U \in \mathcal{U}$ such that $x_o \in U$ and

$$x \in U \Rightarrow \delta < g(x) \qquad (1)$$

By monotonic complete regularity, there exists $f' \in A^{\uparrow}$ such that $f'(x_o) = 1$, $f'(X-U) = 0$, $0 \leq f' \leq 1$. Let $f = \delta f'$. Then

$$x \in X-U \Rightarrow f(x) = 0 ,$$

$$x \in U \Rightarrow f(x) \leq \delta < g(x) \qquad \text{by } (1) .$$

Therefore, $f \leq g$ on X and $f(x_o) = \delta$. Thus $h(x_o) \geq \delta$. Since δ can be chosen arbitrarily close to $g(x_o)$, we have $h(x_o) = g(x_o)$. It follows that

$$g = h = \sup\{f \in A^{\uparrow} : f \leq g\} .$$

Given a non-negative real-valued function f defined on X, and $x_o \in X$, let

$$f_{max}(x_o) = \inf_{L}\left\{\sup_{x}\{f(x) : x \in L\} : x_o \in L \in \mathcal{L}\right\}.$$

Clearly, $f \in A_{\mathcal{L}}$ if and only if $f_{max}(x_o) = f(x_o)$ for every $x_o \in X$. In any case, we have the following.

9.5 $f_{max} \in A_{\mathcal{L}}$ and if $g \in A_{\mathcal{L}}$, $g \geq f$, then $f_{max} \leq g$. Thus f_{max} is the least function in $A_{\mathcal{L}}$ which is greater than or equal to f .

Take arbitrary $c < f_{max}(x_0)$

proof: Let $x_0 \in X$, ~~$f_{max}(x_0) = c$~~ . Given $\varepsilon > 0$, there exists $L \in \mathcal{L}$ such that $x_0 \in L$ and

$$x \in L \Rightarrow f(x) \leq c + \varepsilon . \text{ For } y \in L,\ f_{max}(y) \leq c + \varepsilon .$$

It follows that $f_{max} \in A_{\mathcal{L}}$.

Next, if $g \in A_{\mathcal{L}}$ and $g \geq f$, then

$$f_{max} \leq g_{max} = g ,$$

which completes the proof.

$(X, \mathcal{T}, \leq)$ is said to be extremely order disconnected (in $\mathcal{L}$) if and only if

$$L \in \mathcal{L} \Rightarrow cl_{\mathcal{U}} L \in \mathcal{L} ,$$

where $cl_{\mathcal{U}} L$ is the closure of L in the topology $\mathcal{U}$.

Remark: A dual definition can be formulated for extreme order disconnectedness in $\mathcal{U}$. For simplicity, we consider only the one case, and observe that the next two results have dual formulations.

9.6 If $(X, \mathcal{T}, \leq)$ is extremely order disconnected

e and $f \in A_{\mathcal{U}}$, th~~a~~en $f_{max} \in A_{\mathcal{U}}$.

proof: Suppose that $f_{max} \notin A_{\mathcal{U}}$. There exists $x_0 \in X$ and $\varepsilon > 0$ such that given $U \in \mathcal{U}$ with $x_0 \in U$, there exists $x_u \in U$ such that

$$f_{max}(x_u) \leq c - \varepsilon , \text{ where } c = f_{max}(x_0) .$$

Take $b > c - \varepsilon$. Since $f_{max} \in A_{\mathcal{L}}$, for each x_u there exists L_u with $x_u \in L_u \in \mathcal{L}$ such that

$$x \in L_u \Rightarrow f(x) \le b .$$

Let

$$P = \bigcup L_u$$

where the union is taken over all $U \in \mathcal{U}$ for which $x_o \in U$. Then $P \in \mathcal{L}$ and by the hypothesis, $cl_{\mathcal{U}} P \in \mathcal{L}$. Since $f \in A_{\mathcal{U}}$,

$$f \le b \text{ on } P \Rightarrow f \le b \text{ on } cl_{\mathcal{U}} P .$$

By construction, $x_o \in cl_{\mathcal{U}} P$.
Therefore $f_{max}(x_o) \le b$. Since b can be taken arbitrarily close to $c - \varepsilon$, this is a contradiction.

Therefore $f_{max} \in A_{\mathcal{U}}$.

$A^{\uparrow}$ is said to be <u>bounded complete</u> if and only
its| if every bounded subset of $A^{\uparrow}$ has ~~the~~ supremum in $A^{\uparrow}$.

<u>9.7</u> $(X, \mathcal{T}, \le)$ is extremely order disconnected if and only if $A^{\uparrow}$ is bounded complete.

<u>proof</u>: (a) Suppose that $(X, \mathcal{T}, \le)$ is extremely order disconnected. Let F be a bounded subset of $A^{\uparrow}$ and let

$$g(x) = \sup \{ f(x) : f \in F \} < \infty \quad \text{given } x \in X .$$

By 9.3, $g \in A_{\mathcal{U}}$ and so by 9.6,

$$g_{max} \in A_{\mathcal{U}} .$$

But by 9.5,

$$g_{max} \in A_{\mathcal{L}} .$$

Hence $g_{max} \in A^{\uparrow}$. Further, if $h \in A^{\uparrow}$ and $h \ge g$, then by 9.5 again

$$g_{max} \le h .$$

Therefore g_{max} is the supremum of F in $A^{\uparrow}$.

(b) Suppose that $A^{\uparrow}$ is bounded complete, and that $L \in \mathcal{L}$. Let

$$F = \{f \in A^{\uparrow} : 0 \leq f \leq 1 \text{ and } f(L) = 0\} .$$

By hypothesis, $g = \sup F$ is contained in $A^{\uparrow}$. Given $x_0 \in L$, by monotonic complete regularity there exists $f \in A^{\uparrow}$ such that $0 \leq f \leq 1$ and $f(x_0) = 0$, and $f(X-L) = 1$. Now f is an upper bound for F, so $g(x_0) = 0$. Hence $g(L) = 0$, and by continuity of g, $g(\mathrm{cl}_{\mathcal{U}} L) = 0$.

Now suppose that $y \in X-\mathrm{cl}_{\mathcal{U}} L$. Then $X-\mathrm{cl}_{\mathcal{U}} L \in \mathcal{U}$ and so by monotonic complete regularity, there exists $f \in A^{\uparrow}$ such that $0 \leq f \leq 1$, $f(y) = 1$, and $f(L) = 0$. Therefore $f \in F$ and $g(y) = 1$. Hence $X-\mathrm{cl}_{\mathcal{U}} L = g^{-1}\{1\}$ is closed in $\mathcal{L}$. Therefore $\mathrm{cl}_{\mathcal{U}} L \in \mathcal{L}$, which completes the proof.

For the case of the discrete order on X, the above result was obtained by Stone [15].

CHAPTER II

RINGS OF CONTINUOUS COMPLEX-VALUED FUNCTIONS

Introduction

Gillman and Henriksen in [6] introduced a class of topological spaces called F-spaces , which in the compact case can be characterized by the property that any two disjoint open F_σ sets shall have disjoint closures. They obtained characterizations of an F-space X in terms of algebraic conditions on the ring $C_R(X)$ of continuous real-valued functions defined on X . Our main purpose here is to characterize an F-space X in terms of the ring (or algebra) $C_C(X)$ of continuous complex-valued functions defined on X . In recent years there has been interest in sup-norm algebras of complex-valued functions defined on F-spaces, and the characterizations obtained in 2.2 have applications in this direction.

Various sub-classes of the class of F-spaces are of interest. T-spaces (see §3 for definitions) are characterized in 3.5 as those spaces X for which $C_C(X)$ is alignable. For these spaces, the spectral theory for positive linear operators on $C_C(X)$ obtained in [4] is applicable.

The problem of determining those spaces X for which $C_C(X)$ is Hermite leads to the more general problem of determining those X for which $C_R(X)$ is an H_n-ring. This is investigated in §4 and partial solutions obtained.

In §5 , we discuss the existence of square roots of elements $f \in C_C(X)$, where X is a U-space . In §6 , it is shown that $C_C(X)$ is regular (in the sense of von Neumann) if and only if $C_R(X)$ is regular.

§1 Preliminaries

Let X be a completely regular (Hausdorff) space. Let $C_C(X)$ and $C_R(X)$ denote the sets of all continuous functions on X , which are complex-valued and real-valued respectively. With the operations of point-wise addition, multiplication, and scalar multiplication, the sets $C_C(X)$ and $C_R(X)$ can be regarded as rings, or algebras over C and R respectively. The subrings of all bounded functions in $C_C(X)$ and $C_R(X)$ are denoted by $C_C^*(X)$ and $C_R^*(X)$ respectively.

Given $f \in C_C(X)$ or $C_R(X)$, we define the zero-set $Z(f)$ of f to be the set

$$Z(f) = \{x \in X : f(x) = 0\} .$$

An elementary but important fact is the following. The family of all zero-sets of continuous complex-valued functions coincides with the family of all zero-sets of continuous real-valued functions. This is evident from the relation

$$Z(f) = Z(|f|) \qquad (f \in C_C(X)) .$$

For real-valued functions we also define the following sets.

$$P(f) = \{ x \in X : f(x) > 0 \} ,$$

$$N(f) = \{ x \in X : f(x) < 0 \} .$$

An element $f \in C_C(X)$ (or $C_R(X)$) is called a <u>unit</u> if and only if it has an inverse in $C_C(X)$ (or $C_R(X)$). Evidently f is a unit if and only if $Z(f) = \emptyset$. f is said to be <u>unitary</u> if and only if $|f(x)| = 1$ for all $x \in X$. If f is a unit, then $\frac{f}{|f|}$ is unitary.

Given $f \in C_C(X)$, we denote its real and imaginary parts by f' and f'' respectively and we write

$$f = f' + i f'' \qquad (f' , f'' \in C_R(X)) .$$

The Stone-Čech compactification of X is denoted by βX . For a discussion of the Stone-Čech compactification, see for example, Gillman and Jerison [7] Chapter 6 .

If S is a commutative ring with identity, the ideal generated by n elements $a_1, \dots , a_n$ is denoted by $(a_1, \dots ,a_n)$. An ideal is said to be <u>principal</u> if and only if it is generated by a single element. We note that for $f , g \in C_R(X)$ (or $C_C(X)$),

$$(f,g) = (1) \text{ if and only if } Z(f) \cap Z(g) = \emptyset .$$

For example, if $f , g \in C_C(X)$ and $Z(f) \cap Z(g) = \emptyset$, then

$$\frac{f f^* + g g^*}{|f|^2 + |g|^2} = 1$$

where f^* and g^* are the complex conjugates of f and g .

§2 F-rings and F-spaces

Definition: A commutative ring S with identity is called an F-ring if and only if every finitely generated ideal is principal.

The following definition of an F-space is that given by Gillman and Henriksen in [6].

Definition A completely regular space X is an F-space if and only if $C_R(X)$ is an F-ring.

The main characterizations of F-spaces which were obtained in [6] are summarized below.

2.1 (Gillman and Henriksen)

The following conditions are equivalent.

(a) X is an F-space, that is, $C_R(X)$ is an F-ring.

(b) Given $f, g \in C_R(X)$, $(f,g) = (|f| + |g|)$.

(c) Given $f \in C_R(X)$, $f = k|f|$ for some $k \in C_R(X)$.

(d) Given a zero-set Z of X, every function $\theta \in C_R^*(X-Z)$ has a continuous extension $h \in C_R^*(X)$.

It is shown in [6] that if X is locally compact and σ-compact, then $\beta X-X$ is a compact F-space. For example, the space R^+ of non-negative reals is locally compact and σ-compact. Also βR^+-R^+ is connected. This gives an example of a compact connected F-space.

One of our characterizations is in terms of self-adjointness of ideals. An ideal I of $C_C(X)$ is said to be self-adjoint if given $f \in I$, the complex conjugate f^* is also contained in I .

We also answer a question of Weiss [16] on whether any sup-norm algebra of continuous complex functions can be an F-ring. If X is a compact F-space, then $C_C(X)$ is a sup-norm algebra of continuous complex functions which by 2.2 (a) is also an F-ring.

2.2 The following conditions are equivalent.

(a) $C_C(X)$ is an F-ring .

(b) Given f , $g \in C_C(X)$, $(f,g) = (|f| + |g|)$.

(c) Given $f \in C_C(X)$, there exist $k_1, k_2 \in C_C(X)$ such that $f = k_1|f|$ and $|f| = k_2 f$.

(d) Given a zero-set Z of X , every function $\theta \in C^*_C(X-Z)$ has a continuous extension $h \in C^*_C(X)$.

(e) Each ideal of $C_C(X)$ is self-adjoint.

(f) X is an F-space .

proof: We establish the cycles of implications $(f) \Rightarrow (d) \Rightarrow (c) \Rightarrow (e) \Rightarrow (f)$ and $(a) \Rightarrow (f) \Rightarrow (b) \Rightarrow (a)$.

$(f) \Rightarrow (d)$. Let $\theta = \theta' + i\,\theta'' \in C^*_C(X-Z)$. Then $\theta', \theta'' \in C^*_R(X-Z)$, and by 2.1 (d) , θ' and θ'' have continuous extensions h' and h'' in $C^*_R(X)$. Then $h = h' + i\,h''$ is a continuous extension of θ and belongs to $C^*_C(X)$.

$(d) \Rightarrow (e)$. Given $f \in C_C(X)$, write $Z = Z(f)$. Then

$$\frac{f}{|f|}, \frac{|f|}{f} \in C^*_C(X-Z) .$$

By hypothesis, $\frac{f}{|f|}$, $\frac{|f|}{f}$ have continuous extensions k_1 and k_2 over X . It then follows that

$$f = k_1|f| \quad \text{and} \quad |f| = k_2 f \quad .$$

(c) $\Rightarrow$ (e) . Let I be an ideal of $C_C(X)$, and let $f \in I$. Since f , $f^* \in C_C(X)$, there exist k_1 , $k_2 \in C_C(X)$ such that

$$f^* = k_1|f^*| = k_1|f| \text{ and } |f| = k_2 f \quad .$$

Hence $f^* = k_1 k_2 f$ so that $f^* \in I$.

(e) $\Rightarrow$ (f) . Let $f \in C_R(X)$. Then $f - i|f| \in C_C(X)$. By hypothesis, $f + i|f| = (f - i|f|)^*$ is in the ideal generated by $f - i|f|$. There exists $h = h' + i\, h'' \in C_C(X)$ such that

$$f + i|f| = (h' + i\, h'')(f - i|f|) \quad .$$

Multiplying both sides by $f - i|f|$, *we have*

$$f^2 + |f|^2 = (h' + i\, h'')(f^2 - 2i|f|f - |f|^2) \; .$$

Simplifying and equating real parts, *we have*

$$|f|^2 = f^2 = h''|f|f \quad .$$

It follows that $f = h''|f|$ (and also $|f| = h'' f$) so that X is an F-space.

(a) $\Rightarrow$ (f) . Let $f \in C_R(X)$. By hypothesis, with ideals taken in $C_C(X)$, $(f, |f|) = (d)$ for some $d \in C_C(X)$. Hence there exist $g, h, s, t \in C_C(X)$ such that $f = g\, d$, $|f| = h\, d$, and $d = s\, f + t|f|$. Then $d = (s\, g + t\, h)\, d$. On $P(f) \cup N(f)$, d has no zeros and $s\, g + t\, h = 1$. Writing out real and imaginary parts, we find that on $P(f)$, $g' = h'$ and $g'' = h''$. On $N(f)$, $g' = -h'$ and $g'' = -h''$. Further, on $P(f) \cup N(f)$, $s'g' - s''g'' + t'h' - t''h'' = 1$. Define

$$a_1 = s'g' - s''g'' + t'g' - t''g'' \quad ,$$

$$a_2 = s'h' - s''h'' + t'h' - t''h'' \quad ,$$
$$b_1 = s'g' - s''g'' - t'g' + t''g'' \quad ,$$
$$b_2 = -s'h' + s''h'' + t'h' - t''h'' \quad .$$

Then

$$a_1a_2 = 1 \quad \text{on} \quad P(f) \qquad a_1a_2 \leq 0 \quad \text{on} \quad N(f) \quad ,$$
$$b_1b_2 \leq 0 \quad \text{on} \quad P(f) \qquad b_1b_2 = 1 \quad \text{on} \quad N(f) \quad .$$

Define $\quad k = \max\{a_1a_2, 0\} - \max\{b_1b_2, 0\}$.
Then $k = 1$ on $P(f)$, $k = -1$ on $N(f)$, and $k \in C_R(X)$.
Hence $f = k|f|$. By 2.1(c) , X is an F-space.

(f) $\Rightarrow$ (b) . Given f , $g \in C_C(X)$, it follows from (c) above and 2.1(b) , that with ideals taken in $C_C(X)$,

$$(f,g) = (|f|, |g|) = (|f| + |g|) \quad .$$

(b) $\Rightarrow$ (a) is trivial.

This completes the proof of 2.2 .

When X is a compact F-space, the number of possible subalgebras of $C_C(X)$ is drastically reduced. This question is discussed by Bade and Curtis in [1] .

The number of prime ideals in $C_C(X)$ is also reduced as the following shows.

<u>2.3</u> Let X be a compact F-space, and let I be a prime ideal in $C_C(X)$. Then I is sup-norm dense in the (unique) maximal ideal which contains it.

<u>proof</u>: It follows from 4I of [7] that if I is a prime ideal in $C_C(X)$, then there exists $p \in X$ such that

$$O_p \subsetneq I \subsetneq M_p \quad ,$$

where $O_p = \left\{ f \in C_c(X) : Z(f) \text{ is a neighbourhood of } p \right\}$

$M_p = \left\{ f \in C_c(X) : f(p) = 0 \right\}$,

and M_p is the unique maximal ideal containing I .

Now O_p , and hence I , separates all points of X except p . By 2.2(e) , I is self-adjoint, so by the Stone-Weierstrass theorem for complex algebras without a unit,

/e

$$\text{cl } I = \left\{ f \in C_c(X) : f(p) = 0 \right\} = M_p$$

where the closure is taken in the sup-norm topology.

§3 Hermite rings and T-spaces

Definition : A commutative ring S with identity is called a Hermite ring if and only if it satisfies the condition

T : given $a , b \in S$, there exist $a_1 , b_1 , d \in S$ such that

$$a = a_1 d , \quad b = b_1 d , \quad (a_1, b_1) = (1) .$$

This condition is just the condition that every matrix over S can be reduced to triangular form (see [5] Theorem 3) .

3.1 A Hermite ring is an F-ring .

proof: With a_1 , b_1 , d as in condition T , we have $(a,b) = (d)$.

The next result is due to Kaplansky [8] and shows that the element d of condition T can be replaced by any generator of the ideal (a,b). A proof may be found in [5] Lemma 4, but this is also a special case of 4.2 in §4 which is proved later.

3.2 Let $a, b \in S$. If a_1, b_1, d exist as in condition T, then for all d' with $(a,b) = (d')$, there exist a_1', b_1' such that

$$a = a_1' d', \quad b = b_1' d' \quad \text{and} \quad (a_1', b_1') = (1) .$$

3.3 If S is a Hermite ring, then given $a_1, \dots, a_n, d$ with $(a_1, \dots, a_n) = (d)$, there exist $b_1, \dots, b_n$ such that $a_1 = b_1 d, \dots, a_n = b_n d$ and $(b_1, \dots, b_n) = (1)$.

proof: By 3.2, the result holds for $n = 2$. Suppose it holds for $n = k$ where $k \geq 2$. Take $a_1, \dots, a_k, a_{k+1}, d$ with $(a_1, \dots, a_k, a_{k+1}) = (d)$. Since S is an F-ring (3.1), $(a_1, \dots, a_k) = (d')$ say. By the hypothesis, there exist $b_1, \dots, b_k$ such that $a_1 = b_1 d', \dots, a_k = b_k d'$ and $(b_1, \dots, b_k) = (1)$. Now $(d', a_{k+1}) = (d)$, so by 3.2, there exist b_k', b_{k+1} such that $d' = b_k' d$, $a_{k+1} = b_{k+1} d$ and $(b_k', b_{k+1}) = (1)$. Then $a_1 = b_1 b_k' d, \dots, a_k = b_k b_k' d$, $a_{k+1} = b_{k+1} d$ and $(b_1 b_k', \dots, b_k b_k', b_{k+1}) = (b_k', b_{k+1}) = (1)$. Hence the result is true for $n = k + 1$. By induction, the result is true for all n.

<u>Definition</u>: A completely regular space X is called a <u>T-space</u> if and only if the ring $C_R(X)$ is a Hermite ring.

Since a Hermite ring is an F-ring, it follows that a T-space is an F-space.

It was stated on p.47 that $\beta R^+ - R^+$ is an F-space. In fact it is a T-space, as is shown in [6]. However, there are examples of F-spaces which are not T-spaces.

<u>3.4</u> If X is a T-space, then $C_C(X)$ is a Hermite ring.

<u>proof</u>: Given $f, g \in C_C(X)$, write $f = f' + i f''$, $g = g' + i g''$ with $f', f'', g', g'' \in C_R(X)$. Since $C_R(X)$ is an F-ring, there exists $d \in C_R(X)$ such that

$(f', f'', g', g'') = (d)$, (ideals taken in $C_R(X)$).

By 3.3, there exist $u', u'', v', v'' \in C_R(X)$ with $(u', u'', v', v'') = (1)$ such that

$$f' = u' d, \quad f'' = u'' d, \quad g' = v' d, \quad g'' = v'' d .$$

Then

$$f = (u' + i u'') d, \quad g = (v' + i v'') d$$

with $(u' + i u'', v' + i v'') = (1)$, since $Z(u') \cap Z(u'') \cap Z(v') \cap Z(v'') = \emptyset$. Hence $C_C(X)$ is Hermite.

Whether or not the converse to 3.4 holds, is unsettled. A more general problem is considered in §4.

One of our characterizations of F-spaces is in terms of the existence of rotations on the algebra $C_C(X)$. A <u>rotation</u> on $C_C(X)$ is a linear operator D mapping $C_C(X)$ onto $C_C(X)$ such that

$$|Df| = |f| \qquad \text{for all} \quad f \in C_C(X) .$$

$C_C(X)$ is said to be <u>alignable</u> if and only if given $f_o \in C_C(X)$, there exists a rotation D on $C_C(X)$ such that

$$D|f_o| = f_o$$

<u>Definition</u>: Let S stand for either $C_C(X)$ or $C_R(X)$. Then S is called a <u>U-ring</u> if and only if for each $f \in S$, f and $|f|$ are associates, that is, there exists a unit u of S such that $f = u|f|$.

The equivalence of conditions (b) and (c) below is due to Bonsall.

<u>3.5</u> The following conditions are equivalent.

(a) X is a T-space.

(b) $C_C(X)$ is a U-ring.

(c) $C_C(X)$ is alignable.

<u>proof</u>: (a) $\Rightarrow$ (b) . Let $f \in C_C(X)$ and write $f = f' + i f''$ with f' , $f'' \in C_R(X)$. Since X is an F-space, we have by 2.1(b) ,

$$(f', f'') = (|f'| + |f''|) \qquad \text{(with ideals in } C_R(X) \text{).}$$

Let $Z = Z(|f|) = Z(|f'| + |f''|)$.

Since $|f| = \sqrt{(f'^2 + f''^2)} \leq |f'| + |f''|$ and

$$|f'| + |f''| \leq 2\sqrt{(f'^2 + f''^2)} = 2|f| , \quad \text{we have}$$

$$\frac{|f'| + |f''|}{|f|} , \frac{|f|}{|f'| + |f''|} \in C_R^*(X-Z)$$

By 2.1(d) , these functions have continuous extensions over X . This shows that

$$(|f'| + |f''|) = (|f|) = (f' , f'') .$$

By the hypothesis and 3.2 , there exist u' , $u'' \in$

$C_R(X)$ **with** $(u', u'') = (1)$ such that

$$f' = u'|f| \quad , \quad f'' = u''|f| \;.$$

Then $u = u' + i\,u''$ is a unit of $C_C(X)$ and

$$f = u|f| \;.$$

(b) $\Rightarrow$ (a) . Let f , $g \in C_R(X)$. Then $h = f + i\,g \in C_C(X)$. By the hypothesis, there exists a unit $u \in C_C(X)$ such that

$$h = u|h| \;.$$

Writing $u = u' + i\,u''$ with u' , $u'' \in C_R(X)$ we have $(u', u'') = (1)$. Hence

$$f = u'|h| \;, \quad g = u''|h| \qquad (u', u'') = (1) \;.$$

This shows that X is a T-space.

(b) $\Rightarrow$ (c) . Let $f_0 \in C_C(X)$. There exists a unitary element $u \in C_C(X)$ such that $f_0 = u|f_0|$ ./ For all $f \in C_C(X)$, $|u\,f| = |f|$, so that the operation of multiplication by u is a rotation on $C_C(X)$ with the required property.

If u is a unit and $f_0 = u|f_0|$, then $\frac{u}{|u|}$ is unitary, and $f_0 = \frac{u}{|u_0|}|f_0|$

(c) $\Rightarrow$ (b) . Let $f_0 \in C_C(X)$. By the hypothesis, there exists a rotation D on $C_C(X)$ such that / $D|f_0| = f_0$ ~~$Df_0 = |f_0|$~~ . We show that $D1$ is unitary and that D is the operation of multiplication by $D1$.

Given $x \in X$, let Φ_x and δ_x denote the linear functionals on $C_C(X)$ defined by

$$\Phi_x(f) = (Df)(x) \qquad (f \in C_C(X)) \;,$$
$$\delta_x(f) = f(x) \qquad (f \in C_C(X)) \;.$$

Then
$$|\Phi_x(f)| = |\delta_x(f)| \qquad (f \in C_C(X)) \;.$$

Hence Φ_x and δ_x have identical null spaces. Therefore, there exists $\lambda \in C$ with $|\lambda| = 1$

such that $\Phi_x = \lambda \delta_x$. Then

$$(D1)(x) = \Phi_x(1) = \lambda\delta_x(1) = \lambda ,$$

and

$$(Df)(x) = \Phi_x(f) = \lambda\delta_x(f)$$

$$= ((D1)(x))f(x) \qquad (f \in C_0(X)) ,$$

so that D is the operation of multiplication by D1.

$D1(x) = \lambda$ Finally, $D1(x) \cancel{= 1(x)} = \cancel{1}$ ($x \in X$) , so that D1 is unitary and $(D1)|f_0| = D|f_0| = f_0$.

This completes the proof of 3.5 .

Remark : The notion of an alignable complex lattice was introduced in [4] and a spectral theory developed for certain positive operators defined on the space.

§4 H_n-rings and T_n-spaces

In this section, n denotes an integer greater than or equal to 2 .

Definition: A commutative ring S with identity is called an H_n-ring if and only if it satisfies the condition

H_n : Given $a_1 , \ldots , a_n \in S$, there exist $b_1 , \ldots , b_n , d \in S$ such that $a_1 = b_1 d , \ldots , a_n = b_n d$ and $(b_1, \ldots ,b_n) = (1)$.

For $n = 2$, the above definition reduces to that for a Hermite ring . It follows from 3.3 , that every Hermite ring is an H_n-ring, for $n = 2,3, \ldots$.

<u>4.1</u> An H_n-ring is an F-ring.

<u>proof</u>: Let $a_1, a_2 \in S$. We show that (a_1,a_2) is principal. By the condition H_n, with $a_2 = a_3 = \ldots = a_n$, there exists $d, b_1, \ldots, b_n \in S$, and $s_1, \ldots, s_n \in S$ such that

$a_1 = b_1 d,\ a_2 = b_2 d, \ldots, a_n = b_n d$ and

$s_1 b_1 + s_2 b_2 + \ldots + s_n b_n = 1$. Hence,

$$d = s_1 b_1 d + s_2 b_2 d + \ldots + s_n b_n d$$
$$= s_1 a_1 + (s_2 + \ldots + s_n) a_2 \in (a_1,a_2).$$

Evidently, $a_1, a_2 \in (d)$, so that $(a_1,a_2) = (d)$.

<u>4.2</u> Let n be even, and let $a_1, \ldots, a_n \in S$. If $b_1, \ldots, b_n, d$ exist as in condition H_n, then for all d' with $(a_1, \ldots, a_n) = (d')$, there exists $b_1', \ldots, b_n'$ such that $(b_1', \ldots, b_n') = (1)$ and $a_1 = b_1' d', \ldots, a_n = b_n' d'$.

<u>proof</u>: Write $d = k d'$ and $d' = l d$. For $i = 1, \ldots, \frac{1}{2}n$, choose s_{2i-1}, s_{2i} such that

$$\sum_{i=1}^{\frac{1}{2}n}(s_{2i-1} b_{2i-1} + s_{2i} b_{2i}) = 1 .$$

Define

$$b_{2i-1}' = k\, l\, s_{2i} - s_{2i} + b_{2i-1} k ,$$

$$b'_{2i} = s_{2i-1} - k\, l\, s_{2i-1} + b_{2i} k .$$

Then

$$b_{2i-1}' d' = a_{2i-1} , \quad b_{2i}' d' = a_{2i} . \text{ Also}$$

$$(s_{2i-1}\, l - b_{2i})\, b_{2i-1}' + (s_{2i}\, l + b_{2i-1}) b_{2i}'$$
$$= b_{2i-1} s_{2i-1} + b_{2i} s_{2i}$$

so that

$$\sum_{i=1}^{\frac{1}{2}n} (s_{2i-1}\, 1 - b_{2i})\, b'_{2i-1} \;+\; (s_{2i}\, 1 + b_{2i-1})\, b'_{2i} \;=\; 1$$

$$\text{and} \quad (b'_1, \ldots, b'_n) = (1) \; .$$

A connection between H_n-rings of real-valued functions and H_n-rings of complex-valued functions is given by the following.

<u>4.3</u> Let X be a completely regular space, and let n be even. Then $C_C(X)$ is an H_n-ring if and only if $C_R(X)$ is an H_{2n}-ring.

<u>proof</u>: (a) Suppose that $C_C(X)$ is an H_n-ring. By 4.1 and 2.2(f) , X is an F-space. Let $f'_1, \ldots, f'_n, f''_1, \ldots, f''_n$ be $2n$ functions in $C_R(X)$. Then for $j = 1, \ldots, n$ define

$$f_j = f'_j + i\, f''_j \in C_C(X) \; .$$

By 2.2(b) ,

$$(f_1, \ldots, f_n) = (|f_1| + \ldots + |f_n|) = (h) \quad \text{with } h \geq 0.$$

By 4.2 , there exist $u_j \in C_C(X)$ such that $(u_1, \ldots, u_n) = (1)$ and $f_j = u_j\, h$ for $j = 1, \ldots, n$. Writing $u_j = u'_j + i\, u''_j$ we have

$$f'_j = u'_j\, h \;, \quad f''_j = u''_j\, h \quad \text{for } j = 1, \ldots, n \;,$$

$$\text{and} \quad (u'_1, \ldots, u'_n, u''_1, \ldots, u''_n) = (1) \;,$$

so that $C_R(X)$ is an H_{2n}-ring.

(b) Suppose that $C_R(X)$ is an H_{2n}-ring and that $f_1, \ldots, f_n \in C_C(X)$. Write $f_j = f'_j + i\, f''_j$ for $j = 1, \ldots, n$. Again X is an F-space and by 2.1(b)

$(f'_1, \ldots, f'_n, f''_1, \ldots, f''_n) = (h)$ with $h \geq 0$.
By 4.2, there exist $u'_j, u''_j \in C_R(X)$ such that $(u'_1, \ldots, u'_n, u''_1, \ldots, u''_n) = (1)$ and

$$f'_j = u'_j h, \quad f''_j = u''_j h \quad \text{for } j = 1, \ldots, n.$$

Writing $u_j = u'_j + i\, u''_j$ we have $(u_1, \ldots, u_n) = (1)$ and

$$f_j = u_j h \qquad \text{for } j = 1, \ldots, n.$$

Hence $C_C(X)$ is an H_n-ring.

In particular, $C_C(X)$ is Hermite if and only if $C_R(X)$ is an H_4-ring. Thus, the question of a converse to 3.4 is contained in the following more general problem.

Problem: Does the condition H_n imply the condition H_k, where $2 \leq k \leq n$?

In the other direction we have the following.

4.4 If S is a H_n-ring, then it is an $H_{n+k(n-1)}$-ring for $k = 1, 2, \ldots$.

proof: The proof is by induction on k, using the method of 3.3.

Definition: A completely regular space X is called a T_n-space if and only if it satisfies the condition T_n: given $f_1, \ldots, f_n \in C_R(X)$, there exist $k_1, \ldots, k_n \in C_R(X)$ such that $(k_1, \ldots, k_n) = (1)$ and

$$f_1 = k_1 |f_1|, \ldots, f_n = k_n |f_n|.$$

Every T_n-space is an F-space and a T_n-space is also a T_k-space for any $k \geq n$.

4.5 Let X be a completely regular space and let n be even. If $C_R(X)$ is an H_n-ring, then X is a T_n-space.

proof: We modify the argument in Theorem 3.2 of [6]. Let $f_1, \dots, f_n \in C_R(X)$. Since X is an F-space,

$$(f_1, \dots, f_n) = (|f_1| + \dots + |f_n|) = (h) \quad \text{with} \quad h \geq 0.$$

By the hypothesis and 4.2 , there exist $g_1, \dots, g_n \in C_R(X)$ such that $(g_1, \dots, g_n) = (1)$ and

$f_1 = g_1 h , \dots , f_n = g_n h$. For $i = 1, \dots, n$, $P(f_i) \subseteq P(g_i)$ and $N(f_i) \subseteq N(g_i)$. Moreover $|g_i(x)| \leq 1$ whenever $f_i(x) \neq 0$. We can, if necessary, replace g_i by a function g_i' with $|g_i'(x)| \leq 1$ for all $x \in X$.

Since X is an F-space, we can find functions $s_i \in C_R(X)$ such that $s_i(P(g_i)) = 1$, $s_i(N(g_i)) = 0$. For $i = 1, \dots, n$, choose $m_i \in C_R(X)$ such that $f_i = m_i|f_i|$. Define

$$k_i = s_i \max\{m_i, g_i\} + (1 - s_i) \min\{m_i, g_i\}$$

Then $k_i \in C_R(X)$, $f_i = k_i|f_i|$ and $Z(k_i) \subseteq Z(g_i)$. Since $Z(g_1) \cap \dots \cap Z(g_n) = \emptyset$, we have $Z(k_1) \cap \dots \cap Z(k_n) = \emptyset$. Hence X is a T_n-space.

In the other direction we have the following. For simplicity, we deal only with the case $n = 3$.

<u>4.6</u> If X is a T_3-space, then $C_R(X)$ is an H_3-ring.

<u>proof</u>: Let $f_1, f_2, f_3 \in C_R(X)$, and let k_1, k_2, k_3 be as in the definition of a T_3-space. Let $h = |f_1| + |f_2| + |f_3|$. Since $(|f_1|, |f_2|, |f_3|) = (h)$, there exist $g_2', g_3' \in C_R(X)$ such that $|f_2| = g_2' h$ and $|f_3| = g_3' h$. Define

$$v = \frac{|k_3|\left\{1 - |k_1| - |k_2| + |k_1||k_2| + g_3'(2|k_1| + 2|k_2| - |k_1||k_2|)\right\}}{|k_1| + |k_2| + |k_3|}$$

For $i = 1, 2, 3$ we have $|k_i| = 1$ where $f_i \neq 0$. Also, $g_3' = 0$ where $f_3 = 0$ but $h \neq 0$, and $g_3' = 1$ where $f_1 = f_2 = 0$ but $h \neq 0$. On making these substitutions,

$$v = g_3' \quad \text{where } h \neq 0 .$$

Hence $|f_3| = v\,h$. Observe also that $Z(k_3) \subseteq Z(v)$. Define $g_1' = 1 - g_2' - v$. Then $g_1' h = h - g_2' h - v h = |f_1|$. Note that $g_1' + g_2' = 1$ where $k_3 = 0$.
Now define

$$u_1 = \frac{|k_1|\left\{1 - |k_2| - |k_3| + |k_2||k_3| + g_1'(2|k_2| + 2|k_3| - |k_2||k_3|)\right\}}{|k_1| + |k_2| + |k_3|}$$

$$u_2 = \frac{|k_2|\left\{1 - |k_1| - |k_3| + |k_1||k_3| + g_2'(2|k_1| + 2|k_3| - |k_1||k_3|)\right\}}{|k_1| + |k_2| + |k_3|}$$

As before, we find that $|f_1| = u_1 h$ and $|f_2| = u_2 h$. It follows from the choice of g_1' that $u_1 + u_2 = 1$ where $k_3 = 0$. Note also that $Z(k_1) \subseteq Z(u_1)$ and $Z(k_2) \subseteq Z(u_2)$.

For $i = 1, 2, 3$, let $p_i \in C_R(X)$ satisfy $k_i = p_i|k_i|$ and $|p_i| \leq 1$. Then $f_i = p_i|f_i|$.

We can now define the required functions g_1, g_2, g_3. Let $g_1 = p_1 u_1$, $g_2 = p_2 u_2$, $g_3 = p_3(1 - u_1 - u_2)$. Then

$$g_1 h = p_1 u_1 h = p_1 |f_1| = f_1 \quad ,$$
$$g_2 h = p_2 u_2 h = p_2 |f_2| = f_2 \quad ,$$
$$g_3 h = p_3(1 - u_1 - u_2) h = p_3 |f_3| = f_3 \quad .$$

To show that $(g_1, g_2, g_3) = (1)$, we show that $|g_1| + |g_2| + |g_3|$ is bounded away from zero. Suppose that $|g_1| = |p_1||u_1| \leq \frac{1}{3}$ and $|g_2| = |p_2||u_2| \leq \frac{1}{3}$. We consider the various cases. If $|p_1| = 1$, then $|u_1| \leq \frac{1}{3}$ and if $|p_1| < 1$ then $k_1 = 0$ and $u_1 = 0$. In either case $|u_1| \leq \frac{1}{3}$. Similarly, $|u_2| \leq \frac{1}{3}$. Since $u_1 + u_2 \neq 1$, we have $k_3 \neq 0$ and hence $|p_3| = 1$. Hence $|g_3| = |1 - u_1 - u_2||p_3| \geq \frac{1}{3}$. This completes the proof of 4.6 .

Let X be the Cartesian product of $[-1,1]$ and $[0,\infty)$ with the product topology. It is shown in [6] that βX-X is an F-space but not a T_2-space. We conjecture that βX-X is a T_3-space, in which case $C_R(\beta X\text{-}X)$ would be an H_3-ring but not a Hermite (H_2) ring.

§5 U-spaces and square roots

Definition: A completely regular space X is called a U-space if and only if $C_R(X)$ is a U-ring.

Recall that $C_R(X)$ is a U-ring if and only if given $f \in C_R(X)$, there exists a unit $u \in C_R(X)$ such that $f = u|f|$.

It is shown in [6] that every U-space is disconnected. Since there are connected spaces X for which $C_C(X)$ is a U-ring (see 3.5 (b)), it follows that the condition that $C_R(X)$ be a U-ring is not equivalent to the condition that $C_C(X)$ be a U-ring.

It is shown in [13] p.280-281, that if X is compact and totally disconnected, then each function $f \in C_C(X)$ has a (continuous) square root. The following proof of this result does not depend on compactness.

<u>5.1</u> Let X be a U-space. Then, each $f \in C_C(X)$ can be written in the form

$$f = |f|\, e^{ig}$$

where $g \in C_R(X)$.

<u>proof</u>: Since a U-space is, evidently, a T-space, $C_C(X)$ is a U-ring. Therefore there exists a unitary element $u \in C_C(X)$ such that $f = |f|u$. Hence we need only prove that $u = e^{ig}$ for some $g \in C_R(X)$.

Let $Z = Z(u + |u|)$. Given $x \in X-Z$, let $\theta(x)$ be the real number / in $[-\pi,\pi]$ for which $u(x) = e^{i\theta(x)}$. Then $-\pi < \theta(x) < \pi$ and θ is a well defined function of $X-Z$ into $(-\pi, \pi)$. Let T be the arc of the unit circle consisting of the points $\xi \in C$ such that $|\xi| = 1$, $|\arg \xi| \neq \pi$. The mapping h defined on

$(-\pi, \pi)$ by $h(x) = e^{ix}$ is a homeomorphism of $(-\pi, \pi)$ onto Γ and so h^{-1} is a homeomorphism of Γ onto $(-\pi, \pi)$. Then $\theta = h^{-1} \circ u$ and so is continuous on $X-Z$.

Since $\theta \in C_R^*(X-Z)$ and X is an F-space, by 2.1(d) θ has a continuous extension θ' over X. By hypothesis, there exists a unitary element $v \in C_R(X)$ such that $\theta' = v|\theta'|$.

Define a function g from X into R by

$$g(x) = \theta(x) \qquad x \in X-Z ,$$
$$= \pi v(x) \qquad x \in Z .$$

We then have $u = e^{ig}$. To show that g is continuous on X, it is sufficient to check continuity at a point $x_0 \in \partial Z$, the boundary of Z. Let (x_w) be a net in $X-Z$ which converges to x_0. Then using the method above, it can be shown that $|\theta(x_w)| \to \pi$. Hence

$$\begin{aligned} g(x_w) &= \theta(x_w) \\ &= v(x_w)|\theta(x_w)| \\ &\to v(x_0)\,\pi \end{aligned}$$

and so g is continuous.

<u>Corollary</u>: Each element $f \in C_C(X)$ has a square root. For with $f = |f|e^{ig}$, take $f^{\frac{1}{2}} = |f|^{\frac{1}{2}} e^{\frac{1}{2}ig}$.

§6 Regular rings and P-spaces

<u>Definition</u>: A commutative ring S with identity is said to be <u>regular</u> if and only if given $a \in S$, there

exists $x \in S$ such that $a^2 x = a$.

<u>Definition</u>: A completely regular space X is said to be a <u>P-space</u> if and only if every zero-set Z is open.

It is easy to show that a P-space is a U-space. The following result on P-spaces is proved in [6] .

<u>6.1</u> X is a P-space if and only if $C_R(X)$ is regular.

In terms of the ring $C_C(X)$ we have the following.

<u>6.2</u> X is a P-space if and only if $C_C(X)$ is regular.

<u>proof</u>: (a) . Suppose that $C_C(X)$ is regular. Let $f \in C_R(X)$. There exists $g \in C_C(X)$ such that $f = gf^2$. Let g' be the real part of g . Then $f = g'f^2$, $g' \in C_R(X)$, so that $C_R(X)$ is regular and by 6.1 , X is a P-space.

(b) . Suppose that X is a P-space, and let $f = f' + i f'' \in C_C(X)$. Then $Z(f') \supseteq Z(f'^2 + f''^2)$, $Z(f'') \supseteq Z(f'^2 + f''^2)$. Since $Z(f')$ and $Z(f'')$ are assumed to be open, f' and f'' are multiples of $f'^2 + f''^2$ (see Problem 1D of [5]) . There exist g' , $g'' \in C_R(X)$ such that

$$f' = g'(f'^2 + f''^2) \quad \text{and} \quad f'' = -g''(f'^2 + f''^2) .$$

It can then be verified that

$$f' + i f'' = (g' + i g'')(f' + i f'')^2 .$$

Hence $C_C(X)$ is a regular ring.

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